CENTRAL ASIAN JOURNAL OF THEORETICAL AND APPLIED SCIENCES

Volume: 04 Issue: 05 | May 2023 ISSN: 2660-5317
https://cajotas.centralasianstudies.org

Local and Global Stability of the $L_{1+\epsilon}$- Curvature

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Received 28th Mar 2023, Accepted 29th Apr 2023, Online 26th May 2023

Abstract: *Origin-centered balls only, when $1 \neq (1+\epsilon) > -(2+\epsilon)$, and only for balls when $\epsilon = 0$ is the $L_{1+\epsilon}$ curvature of a smooth, strictly convex body in in $\mathbb{R}^{2+\epsilon}$ known to be constant. Only for origin-symmetric ellipsoids does the $L_{-(2+\epsilon)}$--curvature remain constant if $\epsilon = 0$. Using the global stability result from [5], we demonstrate that for 0, the volume symmetric difference between K and a translation of the unit ball B is nearly zero if the $(K+\epsilon)_{1+\epsilon}$-curvature is approximately constant. Here, we have K shrunk to the same volume of a unit ball, denoted by K. We demonstrate a comparable result for $\epsilon \leq 1$ in the L^2-distance class of origin-symmetric entities. We also demonstrate a local stability conclusion for $-(2+\epsilon) < 1+\epsilon < 0$: Any strictly convex body with 'nearly' constant $L_{1+\epsilon}$- curvature is 'almost' the unit ball, and this neighborhood surrounds the unit ball. Both a global stability result in R2 for $\epsilon = -3/2$ and a local stability result for $\epsilon > 0$ in the Banach-Mazur distance are demonstrated.*

Keywords: $L_{1+\epsilon}$ *curvature function,* $L_{1+\epsilon}$ *Minkowski inequality.*

1. Introduction

A convex body is called compact convex subset of $\mathbb{R}^{(2+\epsilon)}$ and $(2+\epsilon)$-dimensional Euclidean space, with non-empty interior.

The support of the series functions of a convex body K is defined by

$$\sum h_K^m(u_m) := \max_{x\in K} \sum x \cdot u_m,\ \forall u_m \in S^{(1+\epsilon)}.$$

For $K \in \mathcal{F}_0^{2+\epsilon}$ and $(v_m)_K : \partial K \to S^{(1+\epsilon)}$, and let

$$(v_m)_K^{-1} : S^{(1+\epsilon)} \to \mathbb{R}^{2+\epsilon}$$

be the Gauss parameterization of ∂K. In this case, we have

$$\sum h_K(u_m) = \sum u_m \cdot (v_m)_K^{-1}(u_m)$$

The Gauss curvature of $\partial K, \mathcal{K}_K$, and the curvature function of $\partial K, f_K^m$, are related to the support functions of the convex body by.

$$f_K^m = \sum \frac{1}{\mathcal{K}_K \circ (v_m)_K^{-1}} = \sum \frac{\det\left(\nabla_{i,j}^2 h_K^m + g_{ij} h_K^m\right)}{\det\left(g_{ij}\right)}$$

The function $h_K^{(-\epsilon)m} f_K^m$ is called the $(K+\epsilon)_{1+\epsilon}$-curvature function of K.

For $K \in \mathcal{F}_0^{2+\epsilon}$ we define the scale invariant quantity

$$\mathcal{R}_{1+\epsilon}(K) = \max_{S^{(1+\epsilon)}} \left(h_K^{(-\epsilon)m} f_K^m\right) / \min_{S^{(1+\epsilon)}} \left(h_K^{((-\epsilon)m)} f_K^m\right).$$

Which is due to a collective work of Firey, Lutwak, Andrews, Brendle, Choi, and Daskalopoulos [2],[3],[4],[5],[6],[7],[8],[9] :

Theorem. Let $0 > \epsilon > \infty, \epsilon \neq 2+\epsilon$. If $K \in \mathcal{F}_0^{2+\epsilon}$ satisfies then K is the unit ball.

$$h_K^{(-\epsilon)m} f_K^m \equiv 1$$

The relative asymmetry of two convex bodies $K, K+\epsilon$ is defined as

$$\mathcal{A}(K, K+\epsilon) := \inf_{x\in\mathbb{R}^{2+\epsilon}} \frac{V(K\Delta(\lambda(K+\epsilon)+x))}{V(K)},\quad \text{where } \lambda^{2+\epsilon} = \frac{V(K)}{V(K+\epsilon)}$$

And $K\Delta(K+\epsilon) = (K \setminus (K+\epsilon)) \cup ((K+\epsilon) \setminus K)$.

Theorem 1.1. Let $\epsilon \geq 0$. There exists a constant C independent of dimension with the following property. Any $K \in \mathcal{F}_0^{2+\epsilon}$ satisfies

$$\mathcal{A}(\tilde{K}, B) \leq C(2+\epsilon)^{2.5}\left(\mathcal{R}_{1+\epsilon}(K)^{\frac{1}{1+\epsilon}} - 1\right)^{\frac{1}{2}}$$

The $(K+\epsilon)_{(2+\epsilon)}$-Minkowski inequality also allows us to prove the global stability for $0 \le \epsilon \le 1$ in the class of origin-symmetric bodies in the $(K+\epsilon)^2$-distance. The L^2-distance of $K, K+\epsilon$ is defined by

$$\delta_2(K, K+\epsilon) = \left(\frac{1}{\omega_{2+\epsilon}} \int \sum \quad |h_K^m - h_{K+\epsilon}^m|^2 d\sigma\right)^{\frac{1}{2}}$$

Here σ is the spherical Lebesgue measure on $S^{(1+\epsilon)}$, and ω_i is the surface area of the i-dimensional ball.

Theorem 1.2. Let $0 \le \epsilon \le 1$ and $K \in \mathcal{F}^{2+\epsilon}$ be origin-symmetric. There exists an origin-centered ball $B_{1+\epsilon}$ with radius $1 \le 1+\epsilon \le \mathcal{R}_{1+\epsilon}(K)$, such that

$$\delta_2(\tilde{K}, B_{1+\epsilon}) \le D(\tilde{K})(1 - \mathcal{R}_{1+\epsilon}(K)^{-1})^{\frac{1}{2}}$$

Here the diameter of $\tilde{K}, D(\tilde{K})$, satisfies the inequality

$$D(\tilde{K}) \le 2\left(\left(1 + \left(\frac{4\omega_{(1+\epsilon)}}{\omega_{2+\epsilon}}\right)^{\frac{1}{2}}\right)\mathcal{R}_{1+\epsilon}(K)\right)^3$$

For $1+\epsilon \in (-(2+\epsilon), 0)$, we also establish a local stability result. The points $e_{1+\epsilon}$ will be defined in Definition 2.1.

Theorem 1.3. Let $1+\epsilon \in (-(2+\epsilon), 0)$. There exist positive constants γ, δ, depending only on $(2+\epsilon), (1+\epsilon)$ with the following property. If $K \in \mathcal{F}_0^{2+\epsilon}$ with $e_{1+\epsilon}(K) = 0$ satisfies $\sum |h_{\lambda K}^m - 1|_{C^3} \le \delta$ for some $\lambda > 0$, then $\delta_2(\tilde{K}, B) \le \gamma(\mathcal{R}_{1+\epsilon}(K) - 1)$.

Remark 1.4. For the case $\epsilon = 0$, The logarithmic Minkowski inequality in the class of convex bodies with multiple symmetries proven by Böröczky and Kalantzopoulos in [12] has been used to improve the stability of the cone-volume measure by Böröczky and De in [11]. We proved Theorem 1.2, however is independent of the existence of $(K+\epsilon)_{1+\epsilon}$-Minkowski inequality for $0 \le \epsilon < 1$, it is worth pointing out that such an inequality exists in some particular cases: $0 < \epsilon \le 1$ and in the class of origin-symmetric convex bodies in the plane, or in any dimension and in the class of origin-symmetric bodies for $0 < \epsilon < 1$ where $\epsilon > 0$ is some constant depending on $(2+\epsilon)$; see [13], [14], [15], [16].

Let $K \in \mathcal{F}_0^{2+\epsilon}$. The centro-affine curvature of K, H_K, is defined by

$$H_K := \left(h_K^{(1+\epsilon)m} f_K^m\right)^{-1}$$

It is known the key properties of the centro-affine curvature is that $\min H_K$ and $\max H_K$ are invariant under special linear transformation $S(K+\epsilon)(2+\epsilon)$. That is,

$$\min_{S^{1+\epsilon}} H_K = \min_{S^{1+\epsilon}} H_{\ell K}, \max_{S^{1+\epsilon}} H_K = \max_{S^{1+\epsilon}} H_{\ell K}, \forall \ell \in S(K+\epsilon)(2+\epsilon). \qquad (1.1)$$

Pogorelov's remarkable theorem asserts that an origin-centered ellipsoid is a smooth, strictly convex body with constant centro-affine curvature [17], Thm. [18].[19].[20], [21],[18],[22]. Stability versions of this statement include, in the Banach-Mazur distance $d_{\mathcal{BM}}$. For two convex bodies $K, K+\epsilon$ is defined by

$$d_{\mathcal{BMM}}(K, K+\epsilon) = \min\{\lambda \geq 1 : (K-x) \subseteq \ell((K+\epsilon)-y) \subseteq \lambda(K-x),$$
$$\ell \in G(K+\epsilon)(2+\epsilon), x, y \in \mathbb{R}^{2+\epsilon}\}$$

Question 2. Is there an increasing function f^m with $\lim_{\varepsilon\to 0}\sum f^m(\varepsilon) = 0$ with the following property? If $K \in \mathcal{F}_0^{2+\epsilon}$ satisfies

$$\mathcal{R}_{-(2+\epsilon)}(K) = \frac{\max H_K}{\min H_K} \leq 1+\varepsilon$$

then K is $f^m(\varepsilon)$-close to an ellipsoid in the Banach-Mazur distance.

The following theorem gives a positive answer to this question in the plane under no additional assumption.

Theorem 1.5. There exist $\gamma, \delta > 0$ with the following property. If $K \in \mathcal{F}_0^2$ satisfies $\mathcal{R}_{-2}(K) \leq 1+\delta$, then we have

$$(d_{\mathcal{BM}}(K, B) - 1)^4 \leq \gamma(\mathcal{R}_{-2}(K) - 1)$$

If K has its Santaló point at the origin, then

$$(d_{\mathcal{BM}}(K, B) - 1)^4 \leq \gamma\left(\sqrt{\mathcal{R}_{-2}(K)} - 1\right)$$

In this case, we may allow $\delta = \infty$.

$$d_{\mathcal{BM}}(K, B) \leq \sqrt{\mathcal{R}_{-2}(K)}$$

Theorem 1.6. There exist positive numbers γ, δ, depending only on $(2+\epsilon)$ with the following property. Suppose $K \in \mathcal{F}_0^{2+\epsilon}$ has its Santaló point at the origin, and for some $\ell \in G(K+\epsilon)(2+\epsilon)$ we have $\sum|h_{\ell K}^m - 1|_{C^3} \leq \delta$.

2. Background

$$d_{\mathcal{BBM}}(K, B) \leq \gamma\left(\mathcal{R}_{-(2+\epsilon)}(K) - 1\right)^{\frac{1}{3(3+\epsilon)}} + 1$$

A convex body K is said to be of class C_+^2, if its boundary hypersurface is two-times continuously differentiable and the support function is differentiable.

Let $K, K+\epsilon$ be two convex bodies with the origin of $\mathbb{R}^{2+\epsilon}$ in their interiors. We put $(1+\epsilon)\cdot K := (1+\epsilon)^{\frac{1}{1+\epsilon}}K$ and $(1+2\epsilon)\cdot(K+\epsilon) := (1+2\epsilon)^{\frac{1}{1+\epsilon}}(K+\epsilon)$ where $\epsilon > 0$. For $\epsilon \geq 0$, the

$(K+\epsilon)_{1+\epsilon}$-linear combination $(1+\epsilon)\cdot K +_{1+\epsilon} (1+2\epsilon)$. $(K+\epsilon)$ is defined as the convex body whose support function is given by $\left((1+\epsilon)h_K^{(1+\epsilon)m} + (1+2\epsilon)h_{K+\epsilon}^{(1+\epsilon)m}\right)^{\frac{1}{1+\epsilon}}$.

For $K, K+\epsilon \in \mathcal{K}_0^{2+\epsilon}$, the mixed $(K+\epsilon)_{1+\epsilon}$-volume $V_{1+\epsilon}(K, K+\epsilon)$ is defined as the first variation of the usual volume with respect to the $(K+\epsilon)_{1+\epsilon}$-sum:

$$\frac{2+\epsilon}{1+\epsilon} V_{1+\epsilon}(K, K+\epsilon) = \lim_{\varepsilon \to 0^+} \frac{V(K +_{1+\epsilon} \varepsilon \cdot (K+\epsilon)) - V(K)}{\varepsilon}.$$

Aleksandrov, Fenchel and Jessen for $\epsilon = 0$ and Lutwak [7] for $\epsilon > 0$ have shown that there exists a unique Borel measure $S_{1+\epsilon}(K,\cdot)$ on $S^{1+\epsilon}$, $L_{1+\epsilon}$-surface area measure, such that

$$V_{1+\epsilon}(K, K+\epsilon) = \frac{1}{2+\epsilon} \int \sum \quad h_{K+\epsilon}^{(1+\epsilon)m}(u_m)\, dS_{1+\epsilon}(K, u_m)$$

Moreover, $S_{1+\epsilon}(K,\cdot)$ is absolutely continuous with respect to the surface area measure of $K, S(K,\cdot)$, and has the Radon-Nikodym derivative

$$\frac{dS_{1+\epsilon}(K,\cdot)}{dS(K,\cdot)} = \sum h_K^{(-\epsilon)m}(\cdot)$$

The measure $dS_{1+\epsilon,K} = h_K^{(-\epsilon)m} dS_K$ is known as the $L_{1+\epsilon}$-surface area measure. If theboundary of K is C_+^2, then

$$\frac{dS_K}{d\sigma} = \frac{1}{\mathcal{K}_K \circ v_K^{-1}} = f_K^m$$

For $\epsilon > 0$, the L_{1+}-Minkowski inequality states that for convex bodies $K, K+\epsilon$ with the origin in their interiors we have

$$\frac{1}{2+\epsilon} \int \sum \quad h_{K+\epsilon}^{(1+\epsilon)m} dS_{1+\epsilon}(K) \geq V(K)^{\frac{1+\epsilon}{2+\epsilon}} V(K+\epsilon)^{\frac{1+\epsilon}{2+\epsilon}}$$

with equality holds if and only if K and $K+\epsilon$ are dilates (i.e. for some $\lambda > 0, K = \lambda(K+\epsilon)$; see [19]. For $\epsilon = 0$, the same inequality holds for all $K, K+\epsilon \in \mathcal{K}^{2+\epsilon}$, and equality holds if and only if K is homothetic to $(K+\epsilon)$.

The polar body, K^*, of $K \in \mathcal{K}_0^{2+\epsilon}$ is the convex body defined by

$$K^* = \{y \in \mathbb{R}^{2+\epsilon} : x \cdot y \leq 1, \forall x \in K\}$$

All geometric quantities associated with the polar body are furnished by $*$. For $x \in \operatorname{int} K$, let $K^x := (K - x)^*$. The Santaló point of K, denoted by $s = s(K)$, is the unique point in int K such that

$$V(K^s) \leq V(K^x), \ \forall x \in \operatorname{int} K$$

If $K = -K$, then $s(K) = 0$ and $K^* = K^s$.

The Blaschke-Santaló inequality states that

$V(K^s)V(K) \leq V(B)^2$

and equality holds if and only if K is an ellipsoid.

Definition 2.1. The $(K+\epsilon)_{1+\epsilon}$-widths of $K \in \mathcal{K}^n$ are defined as follows.

(1) For $\epsilon > 0$: $\mathcal{E}_{1+\epsilon}(K) = \frac{1}{\omega_{2+\epsilon}} \inf_{x \in \text{int } K} \int h_{K-x}^{1+\epsilon} d\sigma$.

(2) For $\epsilon = -1$: $\varepsilon_0(K) = \frac{1}{\omega_{2+\epsilon}} \sup_{x \in \text{int } K} \int \sum \log h_{K-x}^{m} d\sigma$.

(3) For $0 < \epsilon < 1$: $\mathcal{E}_{1+\epsilon}(K) = \frac{1}{\omega_{2+\epsilon}} \sup_{x \in \text{int } K} \int h_{K-x}^{(1+\epsilon)m} d\sigma$.

(4) For $0 \leq \epsilon < (2+\epsilon)$: $\varepsilon_{1+\epsilon}(K) = \frac{1}{\omega_{2+\epsilon}} \inf_{x \in \text{int } K} \int \sum h_{K-x}^{(1+\epsilon)m} d\sigma$.

Here $\omega_{2+\epsilon} = (2+\epsilon)\kappa_{2+\epsilon} = \int d\sigma$

Here, $e_{1+\epsilon}$ denotes the unique point at which the corresponding sup or inf is attained. The points $e_{1+\epsilon}$ are always in the interior of the convex body; see e.g.[[23], Lem. 3.1]. If K is origin-symmetric, then $e_{1+\epsilon}(K)$ lies at the origin.

For $\epsilon \geq 0$ by the $L_{1+\epsilon}$-Minkowski inequality we have

$$\mathcal{E}_{1+\epsilon}(\tilde{K}) \geq 1 \qquad (2.1)$$

For $0 < \epsilon \leq 2 + \epsilon$ by the Blaschke-Santaló inequality,

$$\mathcal{E}_0(\tilde{K}) \geq 0,\ \mathcal{E}_{1+\epsilon}(\tilde{K}) \leq 1, \qquad (2.2)$$

and equality holds when K is a ball. Moreover, for $\epsilon < 1$ we have

$$\begin{aligned} \varepsilon_{1+\epsilon}(\tilde{K})\varepsilon_{-1+\epsilon}(\tilde{K}) &= \frac{1}{\omega_{2+\epsilon}^2} \int \sum \quad h_{\tilde{K}-e_{1+\epsilon}(\tilde{K})}^{(1+\epsilon)m} d\sigma \int h_{\tilde{K}-e_{-1+\epsilon}(\tilde{K})}^{-(1+\epsilon)m} d\sigma \\ &\geq \frac{1}{\omega_{2+\epsilon}^2} \int \sum \quad h_{\tilde{K}-e_{-1+\epsilon}(\tilde{K})}^{(1+\epsilon)m} d\sigma h_{\tilde{K}-e_{-1+\epsilon}^{-(1+\epsilon)m}(\tilde{K}}^{-(1+2} \geq 1, \end{aligned} \qquad (2.3)$$

where we used the definition of $e_{1+\epsilon}$ in the last line. Therefore we obtain

and the equality holds only for balls.

$$\mathcal{E}_{1+\epsilon}(\tilde{K}) \geq 1 \qquad (2.4)$$

We conclude by remarking that $\mathcal{E}_{1+\epsilon}$ enjoys the second Eojasiewicz-Simon gradient inequality; see [24],[25].

3. Stability of the width functionals

We show the stability of the inequalities (2.1) and (2.2) $(\epsilon \neq -1)$ (see [1]).

Lemma 3.1. Suppose $1+\epsilon \in [-(2+\epsilon),0)$. Let $K \in \mathcal{K}^{2+\epsilon}$ with $V(K)=V(B)$. Then

$$|e_{1+\epsilon}(K)-s(K)|^2 \leq c_0(1-\varepsilon_{1+\epsilon}(K))D(K)^{1+\epsilon}$$

where $0_0^{-1} := \frac{(1+\epsilon)(\epsilon)}{2\omega_{2+\epsilon}} \int (u_m \cdot v_m)^2 d\sigma(u_m) = \frac{(1+\epsilon)(\epsilon)}{2(2+\epsilon)}$ for any vector v_m, and $D(K)$ denotes the diameter of K.

Proof. We may suppose $e_{1+\epsilon}(K) \neq s(K)$. Define $v_m = -\frac{e_{1+\epsilon}(K)-s(K)}{|e_{1+\epsilon}(K)-s(K)|}$ and

$$e(t) = e_{1+\epsilon}(K) + tv_m,\ t \in [0, |e_{1+\epsilon}(K)-s(K)|].$$

Let us denote the support function of $K-e(t)$ by h_t^m and

$$E(t) := \frac{1}{\omega_{2+\epsilon}} \int \sum \quad h_t^{(1+\epsilon)m} d\sigma$$

Note that $E(0)=\mathcal{E}_{1+\epsilon}(K), E'(0)=0$ and the second derivative of E is given by

Due to $h_t^m \leq D(K)$ we obtain

$$E''(t) = \frac{(1+\epsilon)(\epsilon)}{\omega_{2+\epsilon}} \int \sum \quad h_t^{(\epsilon-1)m}(u_m)(u_m \cdot v_m)^2 d\sigma(u_m)$$

$$D(K)^{\epsilon-1}|e_{1+\epsilon}(K)-s(K)|^2 \leq c_0 \left(\frac{1}{\omega_{2+\epsilon}} \int \sum \quad h_{K-s(K)}^{(1+\epsilon)m} d\sigma - \varepsilon_{1+\epsilon}(K)\right)$$

Now the claim follows from the Blaschke-Santaló inequality. We have the following (see [1]).

Theorem 3.2. The following statements hold.

(1) Let $\epsilon \geq 0$. If $\mathcal{E}_{1+\epsilon}(\tilde{K}) \leq 1+\varepsilon$, then

$$\mathcal{A}(\tilde{K},B)^2 \leq C(2+\epsilon)^5\left((1+\varepsilon)^{\frac{1}{1+\epsilon}}-1\right).$$

Here C is a universal constant that does not depend on $(2+\epsilon)$.

(2) Let $1+\epsilon \in (-(2+\epsilon),0)$. If $\mathcal{E}_{1+\epsilon}(\tilde{K}) \geq 1-\varepsilon$, then there exists an origincentered ball of radius $(1+\epsilon), B_{1+\epsilon}$, such that

$$\delta_2\left(\tilde{K}-e_{1+\epsilon}(\tilde{K}),B_{1+\epsilon}\right)\leq\left(2c_1(D(\tilde{K})+(1+\epsilon))^{(3+\epsilon)}\varepsilon\right)^{\frac{1}{2}}+\left(c_0D(\tilde{K})^{1-\epsilon}\varepsilon\right)^{\frac{1}{2}}$$

Moreover, if $\tilde{K}$ is origin-symmetric, then the last term on the right-hand-side can be dropped and $D(\tilde{K})$ can be replaced by $\frac{1}{2}D(\tilde{K})$. Here

$$1\leq(1+\epsilon)\leq(1-\varepsilon)^{\frac{1}{1+\epsilon}},\ c_1:=\max\left\{\frac{2+\epsilon}{(3+2\epsilon)},-\frac{2+\epsilon}{1+\epsilon}\right\}$$

and c_0 is the constant from Lemma 3.1.

Proof.Case $\epsilon\geq0$: Since $\varepsilon_{1+\epsilon}(\tilde{K})\leq1+\varepsilon$, we have

$$\frac{1}{\omega_{2+\epsilon}}\int\sum\quad h_{\tilde{K}}^{m}d\sigma\leq\varepsilon_{1+\epsilon}(\tilde{K})^{\frac{1}{1+\epsilon}}\leq(1+\varepsilon)^{\frac{1}{1+\epsilon}}$$

The refinement of Urysohn's inequality in [10] completes the proof.

Case $-(2+\epsilon)<1+\epsilon<0$: Assume $V(K)=V(B)$. Denote the support function of $K-e_{1+\epsilon}(K)$ by $h_{1+\epsilon}^{m}$ and the support function of $K-s(K)$ by h_s^m. Since $s(K),e_{1+\epsilon}(K)$ are in the interior of K, both h_s^m and $h_{1+\epsilon}^{m}$ are positive functions.

Let us put

By[[20], Thm. 2.2], we have

$$\begin{aligned}f^m=h_s^{(1+\epsilon)m},g=1,\quad &(1+\epsilon)^2=-(2+\epsilon),(1+2\epsilon)=\frac{2+\epsilon}{(3+2\epsilon)},c_1\\ =\quad&\max\{1+\epsilon,1+2\epsilon\}\end{aligned}$$

$$\begin{aligned}&\sum\frac{\int h_s^{(1+\epsilon)m}d\sigma}{\left(\int\frac{1}{h_s^{(2+\epsilon)m}}d\sigma\right)^{-\frac{1+\epsilon}{2+\epsilon}}\omega_{2+\epsilon}^{\frac{3+2\epsilon}{2+\epsilon}}}\\ &\leq1-\frac{1}{c_1}\sum\left|\frac{h_s^{-\left(\frac{2+\epsilon}{2}\right)m}}{\left(\int\frac{1}{h_s^{(2+\epsilon)m}}d\sigma\right)^{\frac{1}{2}}}-\frac{1}{\omega_{2+\epsilon}^{\frac{1}{2}}}\right|_{(K+\epsilon)^2}^2\end{aligned}\qquad(3.1)$$

Due to our assumption,

$$\int\sum\quad(h_s^{1+\epsilon m})d\sigma\geq\int\sum\quad h_{1+\epsilon}^{(1+\epsilon)m}d\sigma\geq\omega_{2+\epsilon}(1-\varepsilon).\qquad(3.2)$$

By the Blaschke-Santaló inequality, we have

$$\int\sum\frac{1}{h_s^{(2+\epsilon)m}}d\sigma\leq\omega_{2+\epsilon}\qquad(3.3)$$

From (3.2),(3.3), it follows that

$$1-\varepsilon \leq \sum \frac{\int \quad h_s^{(1+\epsilon)m} d\sigma}{\left(\int \frac{1}{h_s^{(2+\epsilon)m}} d\sigma\right)^{-\frac{1+\epsilon}{2+\epsilon}} \omega_{2+\epsilon}^{\frac{3+2\epsilon}{2+\epsilon}}} \tag{3.4}$$

$$(1-\varepsilon)\omega_{2+\epsilon} \leq \int \sum \quad h_s^{(1+\epsilon)m} d\sigma \leq \left(\int \sum \quad \frac{1}{h_s^{(2+\epsilon)m}} d\sigma\right)^{-\frac{1+\epsilon}{2+\epsilon}} \omega_{2+\epsilon^{\frac{3+2\epsilon\epsilon}{2+\epsilon}}}$$

Combining (3.1) and (3.4) we obtain

$$\sum \quad \left| h_s^{\left(\frac{2+\epsilon}{2}\right)m} - (1+\epsilon)^{\frac{2+\epsilon}{2}} \right|_{(K+\epsilon)^2}^2 \leq c_1 \omega_{2+\epsilon} D(K)^{2+\epsilon} \varepsilon \tag{3.5}$$

where

$$(1+\epsilon)^{2+\epsilon} := \omega_{2+\epsilon} \left(\int \sum \quad \frac{1}{h_s^{(2+\epsilon)m}} d\sigma\right)^{-1}, \ 1 \leq (1+\epsilon) \leq (1-\varepsilon)^{\frac{1}{1+\epsilon}} \tag{3.6}$$

In view of (3.5) and (3.6) we have

$$\sum \quad |h_s^m - (1+\epsilon)|_{(K+\epsilon)^2}^2 \leq c_1 \omega_{2+\epsilon} \left(D(K)^{\frac{1}{2}} + (1+\epsilon)^{\frac{1}{2}}\right)^2 D(K)^{2+\epsilon} \varepsilon \tag{3.7}$$

If K is origin-symmetric, then $s(K) = e_{1+\epsilon}(K)$ and the proof is complete. Moreover, in this case we could have replaced $D(K)$ by $\frac{1}{2}D(K)$. Otherwise, to bound $\sum | \, h_{1+\epsilon}^m - (1+\epsilon)|_{(K+\epsilon)^2}$, note that by Lemma 3.1 we have

Therefore,

$$|e_{1+\epsilon}(K) - s(K)|^2 \leq c_0 D(K)^{1+\epsilon} \varepsilon$$

$$\sum \quad |h_{1+\epsilon}^m - (1+\epsilon)|_{(K+\epsilon)^2}$$
$$\leq \sum \ |h_s^m - (1+\epsilon)|_{(K+\epsilon)^2} + \omega_{2+\epsilon}^{\frac{1}{2}} |e_{1+\epsilon}(K) - s(K)|$$
$$\leq \left(2c_1 \omega_{2+\epsilon} (D(K) + (1+\epsilon))^{(3+\epsilon)} \varepsilon\right)^{\frac{1}{2}} + (c_0 \omega_{2+\epsilon} D(K)^{1+\epsilon} \varepsilon)^{\frac{1}{2}}.$$

Remark 3.3. The exponent $1/2$ in (1) is sharp; cf. [26]. Moreover, using [[27], Thm. [11].[8].[8]] it is also possible to give a stability result of order $1/(3+\epsilon)$ in (1) for the Hausdorff distance $d_{\mathcal{H}}(\tilde{K} - \text{cent}\,(\tilde{K}), B)$; we leave out the details to the interested reader. By cutting off opposite caps of height ε of the unit ball, one can see that the optimal order cannot be better than 1 in (2).

Theorem 3.4. Suppose K is an origin-symmetric convex body with

$$\mathcal{E}_{-1}(\tilde{K}) \geq 1-\varepsilon \ \text{ for some } \varepsilon \in (0,1)$$

Then there exists an origin-centered ball $B_{1+\epsilon}$ of radius $1 \leq 1+\epsilon \leq (1-\varepsilon)^{-1}$ such that

Moreover, we have

$$\delta_2\left(\tilde{K}, B_{1+\epsilon}\right) \leq D(\tilde{K})\sqrt{\varepsilon}$$

$$\left(\frac{1}{2}D(\tilde{K})\right)^{\frac{1}{3}} \leq \left(1+\left(\frac{4\omega_{1+\epsilon}}{\omega_{2+\epsilon}}\right)^{\frac{1}{2}}\right)\frac{1}{1-\varepsilon}$$

Proof. Set $h^m = h_{\tilde{K}}^m$. We have

$$\sum \frac{\int \frac{1}{h^m} d\sigma}{\left(\int \frac{1}{h^{2m}} d\sigma\right)^{\frac{1}{2}} \omega_{2+\epsilon}^{\frac{1}{2}}} = 1 - \frac{1}{2}\sum \left|\frac{\frac{1}{h^m}}{\left(\int \frac{1}{h^{2m}} d\sigma\right)^{\frac{1}{2}}} - \frac{1}{\omega_{2+\epsilon}^{\frac{1}{2}}}\right|_{(K+\epsilon)^2}^2 .$$

By our assumption and the Blaschke-Santaló inequality,

Therefore,

$$\int \sum \frac{1}{h^m} d\sigma \geq \omega_{2+\epsilon}(1-\varepsilon), \sum \frac{1}{h^{2m}} d\sigma \leq \omega_{2+\epsilon}$$

Combining these inequalities, we obtain

$$1-\varepsilon \leq \sum \frac{\int \frac{1}{h^m} d\sigma}{\left(\int \frac{1}{h^{2m}} d\sigma\right)^{\frac{1}{2}} \omega_{2+\epsilon}^{\frac{1}{2}}}, (1-\varepsilon)\omega_{2+\epsilon} \leq \int \sum \frac{1}{h^m} d\sigma$$

$$\leq \left(\int \sum \frac{1}{h^{2m}} d\sigma\right)^{\frac{1}{2}} \omega_{2+\epsilon}^{\frac{1}{2}}$$

$$\sum |h^m - (1+\epsilon)|_{(K+\epsilon)^2}^2 \leq \omega_{2+\epsilon} D(\tilde{K})^2 \varepsilon$$

where $(1+\epsilon)^2 := \omega_{2+\epsilon}\left(\int \sum \frac{1}{h^{2m}} d\sigma\right)^{-1}$ and $1 \leq 1+\epsilon \leq (1-\varepsilon)^{-1}$.

Next we estimate the diameter from above. Define

$$S = \left\{v_m \in S^{1+\epsilon} : \sum h_{\tilde{K}}^m(v_m) \leq R^{\frac{1}{3}}\right\}$$

where $R := \max h_K^m = h_K^m(u_m)$ for some vector $u_m \in S^{1+\epsilon}$. We may assume $R > 1$. Then by the Blaschke-Santaló inequality we have

$$(1-\varepsilon)\omega_{2+\epsilon} \leq \int_S \sum \frac{1}{h_K^m} d\sigma + \int_{S^c} \sum \frac{1}{h_K^m} d\sigma$$
$$\leq \sum \left(\int_S \frac{1}{h_K^{2m}} d\sigma\right)^{\frac{1}{2}} |S|^{\frac{1}{2}} + \frac{|S^c|}{R^{\frac{1}{3}}}$$
$$\leq (\omega_{2+\epsilon})^{\frac{1}{2}} |S|^{\frac{1}{2}} + \frac{\omega_{2+\epsilon}}{R^{\frac{1}{3}}}$$

Moreover, by convexity we have $\sum h_K^m(v_m) \geq \sum R|u_m \cdot v_m|$ for all $v_m \in S^{1+\epsilon}$. Hence if $v_m \in S$, then $\sum |u_m \cdot v_m| \leq R^{-\frac{2}{3}}$. Now using $\frac{\pi}{2} - \arccos x \leq 2x,\ \forall x \in [0,1]$

we obtain

Therefore,

$$\frac{1}{2}|S| \leq \omega_{1+\epsilon} \int_{\arccos R^{-\frac{2}{3}}}^{\frac{\pi}{2}} \sin^{1+\epsilon} \theta d\theta \leq \frac{2\omega_{1+\epsilon}}{R^{\frac{2}{3}}}$$

We give the proofs of the main theorems (see [27]).

$$1 - \varepsilon \leq \left(1 + \left(\frac{4\omega_{1+\epsilon}}{\omega_{2+\epsilon}}\right)^{\frac{1}{2}}\right) \frac{1}{R^{\frac{1}{3}}}$$

Proof of Theorem 1.1. Suppose $m_0 \leq h_K^{m(\epsilon)} dS_K / d\sigma \leq M$. Therefore by the $L_{1+\epsilon}$- Minkowski inequality,

$$\frac{m_0}{2+\epsilon} \frac{\int h_K^{(1+\epsilon)m} d\sigma}{V(B)^{\frac{1}{2+\epsilon}} V(K)^{\frac{1+\epsilon}{2+\epsilon}}} \leq \frac{1}{2+\epsilon} \sum \frac{\int h_K^{(1+\epsilon)m} h_K^{m(-\epsilon)} dS_K}{V(B)^{\frac{1}{2+\epsilon}} V(K)^{\frac{1+\epsilon}{2+\epsilon}}}$$
$$= \frac{V(K)^{\frac{1}{2+\epsilon}}}{V(B)^{\frac{1}{2+\epsilon}}} \leq M$$
$$\leq \sum \frac{V(B)^{-\frac{1+\epsilon}{2+\epsilon}} \frac{1}{2+\epsilon} \int h_K^{m(-\epsilon)} dS_K}{V(B)^{1-\frac{1+\epsilon}{2+\epsilon}}} \leq M$$

Hence $\mathcal{E}_{1+\epsilon}(\tilde{K}) \leq \mathcal{R}_{1+\epsilon}(\tilde{K})$, and by Theorem 3.2 the proof is complete.

Proof of Theorem 1.2. Assume $m_0 \leq h_K^{m(-\epsilon)} dS_K / d\sigma \leq M$. Then by the $(L)_{2+\epsilon}$- Minkowski inequality for $\epsilon \geq 0$ we have

Therefore,

$$\frac{1}{2+\epsilon}\int \sum \frac{1}{h_K^{m(2\epsilon+1)}} h_K^{m(-\epsilon)} dS_K \geq V(B)$$

Owing to (2.4) for $\epsilon \geq 0$ we have

$$\frac{M}{2+\epsilon} V(K)^{\frac{2\epsilon+1}{2+\epsilon}} \int \sum \frac{1}{h_K^{m(2\epsilon+1)}} d\sigma \geq V(K)^{\frac{1}{2+\epsilon}} V(B). \qquad (3.8)$$

$$V(K) \geq \frac{m_0}{2+\epsilon}\int \sum h_K^{(1+\epsilon)m} d\sigma \geq m_0 V(K)^{\frac{1+\epsilon}{2+\epsilon}} V(B)^{\frac{1}{2+\epsilon}}$$

and hence for $\epsilon \geq -1$,

$$V(K)^{\frac{1}{2+\epsilon}} \geq m_0 V(B)^{\frac{1}{2+\epsilon}} \qquad (3.9)$$

Since $e_{-1}(K) = 0$, in view of (3.8) we obtain $\mathcal{E}_{-1}(\tilde{K}) \geq \mathcal{R}_{1+\epsilon}(K)^{-1}$. The claim follows from Theorem 3.4.

Remark 3.5. It is clear from the proofs of Theorem 1.1 and Theorem 1.2, that if K has only a positive continuous curvature function, then the same conclusions hold.

Remark 3.6. Applying the Blaschke-Santaló inequality to the left-hand side of (3.8), we obtain

This combined with (3.9) yields

$$\left(\frac{V(K)}{V(B)}\right)^{\frac{2\epsilon+1}{2+\epsilon}} \leq M$$

$$m_0 \leq \left(\frac{V(K)}{V(B)}\right)^{\frac{2\epsilon+1}{2+\epsilon}} \leq M$$

Hence in the class of origin-symmetric bodies if $V(K) = V(B)$, then for any $\epsilon \geq -1$ the $(K+\epsilon)_{1+\epsilon}$-curvature function attains the value 1 at some point; see also Question 3.

Proof of Theorem 1.3. Define $\tilde{\mathcal{E}}_{1+\epsilon}: \mathcal{F}_0^{2+\epsilon} \to (0, \infty)$ by

$$\tilde{\varepsilon}_{1+\epsilon}(h_{K+\epsilon}^m) = \left(\int \sum h_{K+\epsilon}^{1+\epsilon m} d\sigma\right)^{\frac{2+\epsilon}{1+\epsilon}} / V(K+\epsilon)$$

By the divergence theorem we have

$$\sum (\operatorname{grad} \tilde{\varepsilon}_{1+\epsilon})(h_K^m) = \sum \frac{h_K^{(\epsilon)m}\left(\int h_K^{1+\epsilon m} d\sigma\right)^{\frac{2+\epsilon}{1+\epsilon}}}{V(K)^2}\left(\frac{(2+\epsilon)V(K)}{\int h_K^{1+\epsilon m} d\sigma} - h_K^{(-\epsilon)m} f_K^m\right)$$

By [25], Sec. 3.13 (ii)] and[[25], p. 80], there exist $c_2,\delta > 0$, such that for any K with $\sum|h_K^m - 1|_{C^3} \leq \delta$, there holds

$$\left|\tilde{\mathcal{E}}_{1+\epsilon}(K) - \tilde{\mathcal{E}}_{1+\epsilon}(B)\right|^{\frac{1}{2}} \leq c_2 \sum \left|\left(\text{grad } \tilde{\mathcal{E}}_{1+\epsilon}\right)(h_K^m)\right|_{(K+\epsilon)^2}$$

Assuming $m_0 \leq h_K^{(-\epsilon)m} f_K^m \leq M$ gives

$$m_0 \leq \frac{(2+\epsilon)V(K)}{\int \sum \quad h_K^{(1+\epsilon)m} d\sigma} \leq M$$

This in turn implies $\left|\mathcal{E}_{1+\epsilon}(\tilde{K})^{\frac{2+\epsilon}{1+\epsilon}} - 1\right| \leq c_3\left(\mathcal{R}_{1+\epsilon}(\tilde{K}) - 1\right)^2$, as well as

$$\varepsilon_{1+\epsilon}(\tilde{K}) \geq \left(1 + c_3\left(\mathcal{R}_{1+\epsilon}(\tilde{K}) - 1\right)^2\right)^{\frac{1+\epsilon}{2+\epsilon}}$$

Due to Theorem 3.2, the proof is complete.

Proof of Theorem 1.5. Suppose $m_0 \leq H_K \leq M$. By [3, Lem. [18],

$$V(K) \geq \frac{\pi}{\sqrt{M}} \qquad (3.10)$$

In fact, the lemma states that if $V(K) = \pi$, then centro-affine curvature at some point attains 1. Therefore, since $V(\sqrt{\pi/V(K)}K) = \pi$, the function $(V(K)/\pi)^2 H_K$ takes the value 1 at some point. Hence using (3.10) and the Hölder inequality we obtain

$$V(K)V(K^S) \geq \sum \frac{\left(\int \quad h_K^m f_K^m H_K^{\frac{1}{3}} d\sigma\right)^3}{4\int \ h_K^m f_K^m d\sigma} \geq m_0 V(K)^2 \geq \pi^2 \frac{m_0}{M}$$

If the Santaló point is at the origin, then we can obtain a slightly better lower bound for the volume product. By [28], we have

$$\sum \quad H_K(u_m) H_{K^*}(u_m^*) = 1$$

where u_m and u_m^* are related by $\Sigma\left\langle v_K^{-1}(u_m), v_{K^*}^{-1}(u_m^*)\right\rangle = 1$. Since $K^S = K^*$, this yields

$$\frac{1}{M} \leq H_{K^s} \leq \frac{1}{m_0},\ V(K^s) \geq \pi\sqrt{m_0}$$

Therefore, $V(K)V(K^s) \geq \pi^2 \sqrt{\frac{m_0}{M}}$. Now in both cases, the result follows from [29]. The third claim is exactly [29], Cor. [9].

Question 3. Given the previous argument, we would like to raise a question. Suppose $K \in \mathcal{F}_0^{2+\epsilon}, \epsilon \geq 0$, and $V(K) = V(B)$. Is it true that the centro-affine curvature of K attains the value 1 at some point?

Proof of Theorem 1.6. For all $\ell \in G(K+\epsilon)(2+\epsilon)$, we have

$$s(\ell K) = \ell s(K) = 0,\ d_{BM}(\ell K, B) = d_{BM}(K, B).$$

Thus we may assume without loss of generality that

for some $\delta > 0$ to be determined.

$$\sum \quad |h_K^m - 1|_{C^3} \leq \delta$$

Define the functional $\mathcal{P}: \mathcal{F}_0^{2+\epsilon} \to (0, \infty)$ by

We have

$$\mathcal{P}(K+\epsilon) = \mathcal{P}(h_{K+\epsilon}^m) = \frac{1}{V(K+\epsilon)V((K+\epsilon)^*)}$$

$$\begin{aligned} \sum \quad (\mathrm{grad}\ \mathcal{P})(h_K^m) &= \sum\ \mathcal{P}^2(K)\left(\frac{V(K)}{h_K^{((2+\epsilon)+1)m}} - V(K^*)f_K^m\right) \\ &= \sum\ \frac{V(K^*)\mathcal{P}^2(K)}{h_K^{(1+\epsilon)m}}\left(\frac{V(K)}{V(K^*)} - \frac{1}{H_K}\right) \end{aligned} \qquad (3.11)$$

By[[25], Sec. 3.13 (ii)], there exist $\delta, c_2 > 0$ and $\alpha \in (0, 1/2]$, such that for any K with $\sum |h_K^m - 1|_{C^3} \leq \delta$, we have

$$\left|\frac{1}{V(K)V(K^*)} - \frac{1}{V(B)^2}\right|^{1-\alpha} \leq c_2 \sum\ |(\mathrm{grad}\ \mathcal{P})(h_K^m)|_{(K+\epsilon)^2} \qquad (3.12)$$

By[[25], p. 80] and[[30], Lem. 4.1, 4.2] we can choose $\alpha = 1/2$.

We estimate the right-hand side of (3.12). Note that $m_0 \leq H_K \leq M$ implies that

Therefore we obtain

$$\frac{1}{M} \leq \frac{V(K)}{V(K^*)} = \sum \frac{\int \quad h_K^m f_K^m d\sigma}{\int \ h_K^m f_K^m H_K d\sigma} \leq \frac{1}{m_0}$$

$$\frac{1}{M} \leq \frac{V(K)}{V(K^*)} \leq \frac{1}{m_0} \quad \text{and} \quad \left|\frac{V(K)}{V(K^*)} - \frac{1}{H_K}\right| \leq \frac{M - m_0}{M m_0} \qquad (3.13)$$

On the other hand, by (3.13) and the Blaschke-Santaló inequality,

$$V(K^*)^2 \leq M V(B)^2 \qquad (3.14)$$

Putting (3.11),(3.12),(3.13), and (3.14) all together we arrive at

$$\left|\frac{1}{V(K)V(K^*)} - \frac{1}{V(B)^2}\right|^{\frac{1}{2}} \leq c_3\left(\mathcal{R}_{-(2+\epsilon)}(K) - 1\right)\sum \frac{\mathcal{P}^2(K)\left|h_K^{-m(1+\epsilon)}\right|_{(K+\epsilon)^2}}{V(K^*)}$$

Since we are in a small neighborhood of the unit ball, the term

$$\sum \frac{\mathcal{P}^2(K)\left|h_K^{-m(1+\epsilon)}\right|_{(K+\epsilon)^2}}{V(K^*)}$$

is bounded. Using again the Blaschke-Santaló inequality we obtain

$$1 - c_4\left(\mathcal{R}_{-(2+\epsilon)}(K) - 1\right)^2 \leq \frac{V(K)V(K^*)}{V(B)^2}$$

In view of [[19], Thm. 1.1], the proof is complete.

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