Local and Global Stability of the $L_{1+\epsilon}$-Curvature

Salih Yousuf Mohamed Salih
Department of mathematics, Faculty of Science, Bakht Al-Ruda University, Duaim.
salih7175.ss@gamil.com

Shahinaz.A. Elsamani
Department of mathematics, Faculty of Science, Bakht Al-Ruda University, Duaim.
Shahinazel121@gmail.com

Abstract: Origin-centered balls only, when $1 \neq (1 + \epsilon) > -(2 + \epsilon)$, and only for balls when $\epsilon = 0$ is the $L_{1+\epsilon}$-curvature of a smooth, strictly convex body in in $\mathbb{R}^{2+\epsilon}$ known to be constant. Only for origin-symmetric ellipsoids does the $L_{-(2+\epsilon)}$-curvature remain constant if $\epsilon = 0$. Using the global stability result from [5], we demonstrate that for 0, the volume symmetric difference between $K$ and a translation of the unit ball $B$ is nearly zero if the $(K + \epsilon)_{1+\epsilon}$-curvature is approximately constant. Here, we have $K$ shrunk to the same volume of a unit ball, denoted by $K$. We demonstrate a comparable result for $\epsilon \leq 1$ in the $L^2$-distance class of origin-symmetric entities. We also demonstrate a local stability conclusion for $-(2 + \epsilon) < 1 + \epsilon < 0$: Any strictly convex body with 'nearly' constant $L_{1+\epsilon}$-curvature is 'almost' the unit ball, and this neighborhood surrounds the unit ball. Both a global stability result in $\mathbb{R}^2$ for $\epsilon = -3/2$ and a local stability result for $\epsilon > 0$ in the Banach-Mazur distance are demonstrated.

Keywords: $L_{1+\epsilon}$ curvature function, $L_{1+\epsilon}$ Minkowski inequality.

1. Introduction

A convex body is called compact convex subset of $\mathbb{R}^{(2+\epsilon)}$ and $(2 + \epsilon)$-dimensional Euclidean space, with non-empty interior.
The support of the series functions of a convex body $K$ is defined by

$$\sum \hat{h}_K^m(u_m) := \max_{x \in K} \sum x \cdot u_m, \forall u_m \in S^{2+\varepsilon}. $$

For $K \in \mathcal{F}_0^{2+\varepsilon}$ and $(v_m)_K : \partial K \to S^{(1+\varepsilon)}$, and let

$$(v_m)_K^{-1} : S^{(1+\varepsilon)} \to \mathbb{R}^{2+\varepsilon}$$

be the Gauss parameterization of $\partial K$. In this case, we have

$$\sum \hat{h}_K^m(u_m) = \sum u_m \cdot (v_m)_K^{-1}(u_m)$$

The Gauss curvature of $\partial K, K^h$, and the curvature function of $\partial K, f^m_K$, are related to the support functions of the convex body by.

$$f^m_K = \sum \frac{1}{\mathcal{K}_K \circ (v_m)_K^{-1}} \sum \frac{|\det (\nabla_{ij} h_K^m + g_{ij} h_K^m)|}{\det (g_{ij})}$$

The function $h_K^{(-\varepsilon)m,f^m_K}$ is called the $(K + \varepsilon)_{1+\varepsilon}$-curvature function of $K$.

For $K \in \mathcal{F}_0^{2+\varepsilon}$ we define the scale invariant quantity

$$\mathcal{R}_{1+\varepsilon}(K) = \max_{S^{(1+\varepsilon)}} \left( h_{(K + \varepsilon)_{1+\varepsilon}}^{(-\varepsilon)m,f^m_K} \right) / \min_{S^{(1+\varepsilon)}} \left( h_{(K + \varepsilon)_{1+\varepsilon}}^{(-\varepsilon)m,f^m_K} \right).$$

Which is due to a collective work of Firey, Lutwak, Andrews, Brendle, Choi, and Daskalopoulos [2],[3],[4],[5],[6],[7],[8],[9] :

Theorem. Let $0 > \varepsilon > \infty, \varepsilon \neq 2 + \varepsilon$. If $K \in \mathcal{F}_0^{2+\varepsilon}$ satisfies then $K$ is the unit ball.

$$h_{(K + \varepsilon)_{1+\varepsilon}}^{(-\varepsilon)m,f^m_K} \equiv 1$$

The relative asymmetry of two convex bodies $K, K + \varepsilon$ is defined as

$$\mathcal{A}(K, K + \varepsilon) := \inf_{x \in S^{2+\varepsilon}} \frac{V(K, \lambda(K + \varepsilon) + x)}{V(K)}, \text{ where } \lambda^{2+\varepsilon} = \frac{V(K)}{V(K + \varepsilon)}$$

And $K, \lambda(K + \varepsilon) = (K \setminus (K + \varepsilon)) \cup ((K + \varepsilon) \setminus K)$.

**Theorem 1.1.** Let $\varepsilon \geq 0$. There exists a constant $C$ independent of dimension with the following property. Any $K \in \mathcal{F}_0^{2+\varepsilon}$ satisfies

$$\mathcal{A}(K, B) \leq C(2 + \varepsilon)^{2.5} \left( \mathcal{R}_{1+\varepsilon}(K)^{\frac{1}{1+\varepsilon}} - 1 \right)^{\frac{1}{3}}$$
The \((K + \varepsilon)_{(2+\varepsilon)}\)-Minkowski inequality also allows us to prove the global stability for \(0 \leq \varepsilon \leq 1\) in the class of origin-symmetric bodies in the \((K + \varepsilon)^2\)-distance. The \(L^2\)-distance of \(K, K + \varepsilon\) is defined by

\[
\delta_2(K, K + \varepsilon) = \left(\frac{1}{\omega_{2+\varepsilon}} \int \sum |h_K^{\varepsilon} - h_{K+\varepsilon}^{\varepsilon}|^2 \, d\sigma\right)^{\frac{1}{2}}
\]

Here \(\sigma\) is the spherical Lebesgue measure on \(S^{(1+\varepsilon)}\), and \(\omega_i\) is the surface area of the \(i\)-dimensional ball.

**Theorem 1.2.** Let \(0 \leq \varepsilon \leq 1\) and \(\vec{K} \in \mathcal{F}^{2+\varepsilon}\) be origin-symmetric. There exists an origin-centered ball \(B_{1+\varepsilon}\) with radius \(1 \leq 1 + \varepsilon \leq \mathcal{R}_{1+\varepsilon}(\vec{K})\), such that

\[
\delta_2(\vec{K}, B_{1+\varepsilon}) \leq \mathcal{D}(\vec{K}) \left(1 - \mathcal{R}_{1+\varepsilon}(\vec{K})^{-1}\right)^{\frac{3}{2}}
\]

Here the diameter of \(\vec{K}\), \(\mathcal{D}(\vec{K})\), satisfies the inequality

\[
\mathcal{D}(\vec{K}) \leq 2 \left(1 + \left(\frac{4\omega_{[1+\varepsilon]}}{\omega_{2+\varepsilon}}\right)\right)^{\frac{1}{2}} \mathcal{R}_{1+\varepsilon}(\vec{K})^{\frac{3}{2}}
\]

For \(1 + \varepsilon \in (-2 + \varepsilon, 0)\), we also establish a local stability result. The points \(\theta_{1+\varepsilon}\) will be defined in Definition 2.1.

**Theorem 1.3.** Let \(1 + \varepsilon \in (-2 + \varepsilon, 0)\). There exist positive constants \(\gamma, \delta\), depending only on \((2 + \varepsilon), (1 + \varepsilon)\) with the following property. If \(K \in \mathcal{F}^{2+\varepsilon}\) with \(\theta_{1+\varepsilon}(K) = 0\) satisfies \(\sum |h_K^{\varepsilon} - 1| \leq \delta\) for some \(\lambda > 0\), then \(\delta_2(\vec{K}, B) \leq \gamma (\mathcal{R}_{1+\varepsilon}(K) - 1)\).

**Remark 1.4.** For the case \(\varepsilon = 0\). The logarithmic Minkowski inequality in the class of convex bodies with multiple symmetries proven by Böröczky and Kalantzopoulos in [12] has been used to improve the stability of the cone-volume measure by Böröczky and De in [11]. We proved Theorem 1.2, however is independent of the existence of \((K + \varepsilon)_{(2+\varepsilon)}\)-Minkowski inequality for \(0 \leq \varepsilon \leq 1\), it is worth pointing out that such an inequality exists in some particular cases: \(0 < \varepsilon \leq 1\) and in the class of origin-symmetric convex bodies in the plane, or in any dimension and in the class of origin-symmetric bodies for \(0 < \varepsilon < 1\) where \(\varepsilon > 0\) is some constant depending on \((2 + \varepsilon)\); see [13],[14],[15],[16].

Let \(K \in \mathcal{F}^{2+\varepsilon}\). The centro-affine curvature of \(K, H_K\), is defined by

\[
H_K^{\varepsilon} = \left(\frac{h_K^{(1+\varepsilon)m}}{h_K^{m}}\right)^{-1}
\]

It is known the key properties of the centro-affine curvature is that \(\min H_K\) and \(\max H_K\) are invariant under special linear transformation \(S(K + \varepsilon)(2 + \varepsilon)\). That is,

\[
\min H_K^{\varepsilon} = \min H_K^{\ell}, \quad \max H_K^{\varepsilon} = \max H_K^{\ell}, \quad \forall \ell \in S(K + \varepsilon)(2 + \varepsilon).
\]

(1.1)
Pogorelov's remarkable theorem asserts that an origin-centered ellipsoid is a smooth, strictly convex body with constant centro-affine curvature \([17]\), Thm. \([18],[19],[20],[21],[18],[22]\). Stability versions of this statement include, in the Banach-Mazur distance \(d_{\text{BM}}\). For two convex bodies \(K, K + \varepsilon\) is defined by

\[
d_{\text{BM}}(K, K + \varepsilon) = \min \{ \lambda \geq 1; (K - x) \subseteq \ell((K + \varepsilon) - y) \subseteq \lambda(K - x), \ell \in G(K + \varepsilon)(2 + \varepsilon), x, y \in \mathbb{R}^{2+\varepsilon} \}
\]

Question 2. Is there an increasing function \(f^m\) with \(\lim_{\varepsilon \to 0} \sum f^m(\varepsilon) = 0\) with the following property? If \(K \in F_0^{2+\varepsilon}\) satisfies

\[
\mathcal{R}_{-(2+\varepsilon)}(K) = \frac{\max H_K}{\min H_K} \leq 1 + \varepsilon
\]

then \(K\) is \(f^m(\varepsilon)\)-close to an ellipsoid in the Banach-Mazur distance.

The following theorem gives a positive answer to this question in the plane under no additional assumption.

**Theorem 1.5.** There exist \(\gamma, \delta > 0\) with the following property. If \(K \in F_0^2\) satisfies \(\mathcal{R}_{-2}(K) \leq 1 + \delta\), then we have

\[
(d_{\text{BM}}(K, B) - 1)^4 \leq \gamma(\mathcal{R}_{-2}(K) - 1)
\]

If \(K\) has its Santaló point at the origin, then

\[
(d_{\text{BM}}(K, B) - 1)^4 \leq \gamma(\sqrt{\mathcal{R}_{-2}(K)} - 1)
\]

In this case, we may allow \(\delta = \infty\).

\[
d_{\text{BM}}(K, B) \leq \sqrt{\mathcal{R}_{-2}(K)}
\]

**Theorem 1.6.** There exist positive numbers \(\gamma, \delta\), depending only on \((2 + \varepsilon)\) with the following property. Suppose \(K \in F_0^{2+\varepsilon}\) has its Santaló point at the origin, and for some \(\ell \in G(K + \varepsilon)(2 + \varepsilon)\) we have

\[
\sum |h_{K, \ell}^m - 1|_{C^2} \leq \delta.
\]

2. **Background**

\[
d_{\text{BM}}(K, B) \leq \gamma(\mathcal{R}_{-(2+\varepsilon)}(K) - 1)^{\frac{1}{2(2+\varepsilon)}} + 1
\]

A convex body \(K\) is said to be of class \(C_{+}^2\), if its boundary hypersurface is two-times continuously differentiable and the support function is differentiable.

Let \(K, K + \varepsilon\) be two convex bodies with the origin of \(\mathbb{R}^{2+\varepsilon}\) in their interiors. We put \((1 + \varepsilon) \cdot K = (1 + \varepsilon)^{1+\varepsilon} K\) and \((1 + 2\varepsilon) \cdot (K + \varepsilon) = (1 + 2\varepsilon)^{1+\varepsilon}(K + \varepsilon)\) where \(\varepsilon > 0\). For \(\varepsilon \geq 0\), the
linear combination \((1 + \epsilon) \cdot K + 2\epsilon \cdot (K + \epsilon)\) is defined as the convex body whose support function is given by 
\[
((1 + \epsilon)h_{K}^{(1+\epsilon)m} + (1 + 2\epsilon)h_{K+\epsilon}^{(1+\epsilon)m})^{\frac{1}{1+\epsilon}}.
\]

For \(K, K + \epsilon \in K_{0}^{2+\epsilon}\), the mixed \((K + \epsilon)_{1+\epsilon}\)-volume \(V_{1+\epsilon}(K, K + \epsilon)\) is defined as the first variation of the usual volume with respect to the \((K + \epsilon)_{1+\epsilon}\)-sum:

\[
\frac{2 + \epsilon}{1 + \epsilon} V_{1+\epsilon}(K, K + \epsilon) = \lim_{\epsilon \to 0^+} \frac{V(K + (1+\epsilon) \cdot (K + \epsilon)) - V(K)}{\epsilon}.
\]

Aleksandrov, Fenchel and Jessen for \(\epsilon = 0\) and Lutwak [7] for \(\epsilon > 0\) have shown that there exists a unique Borel measure \(S_{1+\epsilon}(K, r)\) on \(S^{1+\epsilon}, L_{1+\epsilon}\)-surface area measure, such that

\[
V_{1+\epsilon}(K, K + \epsilon) = \frac{1}{2 + \epsilon} \int \sum h_{K+\epsilon}^{(1+\epsilon)m}(u_{m}) dS_{1+\epsilon}(K, u_{m})
\]

Moreover, \(S_{1+\epsilon}(K, r)\) is absolutely continuous with respect to the surface area measure of \(K, S(K, r)\), and has the Radon-Nikodym derivative

\[
\frac{dS_{1+\epsilon}(K, r)}{dS(K, r)} = \sum h_{K}^{(-\epsilon)m}(\cdot)
\]

The measure \(dS_{1+\epsilon} = h_{K}^{(-\epsilon)m} dS_{K}\) is known as the \(L_{1+\epsilon}\)-surface area measure. If the boundary of \(K\) is \(C^{2}_{0}\), then

\[
\frac{dS_{K}}{d\sigma} = \frac{1}{K_{0} \circ V_{K}^{-1}} = f_{K}^{m}
\]

For \(\epsilon > 0\), the \(L_{1+}\)-Minkowski inequality states that for convex bodies \(K, K + \epsilon\) with the origin in their interiors we have

\[
\frac{1}{2 + \epsilon} \int \sum h_{K+\epsilon}^{(1+\epsilon)m} dS_{1+\epsilon}(K) \geq V(K)^{\frac{1+\epsilon}{2+\epsilon}} V(K + \epsilon)^{\frac{1+\epsilon}{2+\epsilon}}
\]

with equality holds if and only if \(K\) and \(K + \epsilon\) are dilates (i.e. for some \( \lambda > 0, K = \lambda(K + \epsilon)\); see [19]. For \(\epsilon = 0\), the same inequality holds for all \(K, K + \epsilon \in K^{2+\epsilon}\), and equality holds if and only if \(K\) is homothetic to \((K + \epsilon)\).

The polar body, \(K^*\), of \(K \in K_{0}^{2+\epsilon}\) is the convex body defined by

\[
K^* = \{y \in \mathbb{R}^{2+\epsilon} : x \cdot y \leq 1, \forall x \in K\}
\]

All geometric quantities associated with the polar body are furnished by \(*. For x \in \text{int } K, let K_x = (K - x)^*\). The Santaló point of \(K\), denoted by \(s = s(K)\), is the unique point in \(\text{int } K\) such that

\[
V(K^*) \leq V(K^s), \forall x \in \text{int } K
\]
If \( K = -K \), then \( s(K) = 0 \) and \( K^* = K^s \).

The Blaschke-Santaló inequality states that
\[ V(K^*) V(K) \leq V(B)^2 \]
and equality holds if and only if \( K \) is an ellipsoid.

**Definition 2.1.** The \( (K + \varepsilon)_{1+\varepsilon} \)-widths of \( K \in \mathcal{K}^n \) are defined as follows.

1. For \( \varepsilon > 0 \): \( \mathcal{E}_{1+\varepsilon}(K) = \frac{1}{\omega_{n+\varepsilon}} \inf_{x \in \text{int} K} \int h^{1+\varepsilon}_{K-x} d\sigma. \)
2. For \( \varepsilon = -1 \): \( \mathcal{E}_0(K) = \frac{1}{\omega_{n+\varepsilon}} \sup_{x \in \text{int} K} \int \log h^m_{K-x} d\sigma. \)
3. For \( 0 < \varepsilon < 1 \): \( \mathcal{E}_{1+\varepsilon}(K) = \frac{1}{\omega_{n+\varepsilon}} \sup_{x \in \text{int} K} \int h^{(1+\varepsilon)m}_{K-x} d\sigma. \)
4. For \( 0 \leq \varepsilon < (2 + \varepsilon) \): \( \mathcal{E}_{1+\varepsilon}(K) = \frac{1}{\omega_{n+\varepsilon}} \inf_{x \in \text{int} K} \int \sum h^{(1+\varepsilon)m}_{K-x} d\sigma. \)

Here \( \omega_{2+\varepsilon} = (2 + \varepsilon) \omega_{2+\varepsilon} = \int d\sigma \)

Here, \( e_{1+\varepsilon} \) denotes the unique point at which the corresponding sup or inf is attained. The points \( e_{1+\varepsilon} \) are always in the interior of the convex body; see e.g. [23], Lem. 3.1. If \( K \) is origin-symmetric, then \( e_{1+\varepsilon}(K) \) lies at the origin.

For \( \varepsilon \geq 0 \) by the \( L_{1+\varepsilon} \)-Minkowski inequality we have
\[ \mathcal{E}_{1+\varepsilon}(K) \geq 1 \quad (2.1) \]

For \( 0 < \varepsilon \leq 2 + \varepsilon \) by the Blaschke-Santaló inequality,
\[ \mathcal{E}_0(K) \geq 0, \mathcal{E}_{1+\varepsilon}(K) \leq 1, \quad (2.2) \]
and equality holds when \( K \) is a ball. Moreover, for \( \varepsilon < 1 \) we have
\[ \mathcal{E}_{1+\varepsilon}(K) \geq \frac{1}{\omega_{2+\varepsilon}} \int \sum h^{(1+\varepsilon)m}_{K-x(4+\varepsilon)} d\sigma \int h^{-(1+\varepsilon)m}_{K-x(4+\varepsilon)} d\sigma \geq 1, \quad (2.3) \]
where we used the definition of \( e_{1+\varepsilon} \) in the last line. Therefore we obtain
and the equality holds only for balls.
\[ \mathcal{E}_{1+\varepsilon}(K) \geq 1 \quad (2.4) \]
3. Stability of the width functionals

We show the stability of the inequalities (2.1) and (2.2) ($\varepsilon \neq -1$) (see [1]).

Lemma 3.1. Suppose $1 + \varepsilon \in \mathbb{R}$. Let $K \in \mathcal{K}^{2+\varepsilon}$ with $V(K) = V(B)$. Then

$$|e_{1+\varepsilon}(K) - s(K)|^2 \leq c_0(1 - e_{1+\varepsilon}(K))D(K)^{1+\varepsilon}$$

where $0^{-1}_\omega = \frac{(1+\varepsilon)(\varepsilon)}{2\omega_{1+\varepsilon}} \int (u_m \cdot v_m)^2 d\sigma(u_m) = \frac{(1+\varepsilon)(\varepsilon)}{2\omega_{2+\varepsilon}}$ for any vector $v_m$, and $D(K)$ denotes the diameter of $K$.

Proof. We may suppose $e_{1+\varepsilon}(K) \neq s(K)$. Define $v_m = -\frac{e_{1+\varepsilon}(K) - s(K)}{|e_{1+\varepsilon}(K) - s(K)|}$ and

$$e(t) = e_{1+\varepsilon}(K) + tv_m, \quad t \in [0, |e_{1+\varepsilon}(K) - s(K)|].$$

Let us denote the support function of $K - e(t)$ by $h_{tm}^m$ and

$$E(t) = \frac{1}{\omega_{2+\varepsilon}} \int \sum h_{tm}^{(1+\varepsilon)m} d\sigma$$

Note that $E(0) = e_{1+\varepsilon}(K), E'(0) = 0$ and the second derivative of $E$ is given by

Due to $h_{tm}^m \leq D(K)$ we obtain

$$E''(t) = \frac{(1+\varepsilon)(\varepsilon)}{2\omega_{1+\varepsilon}} \int \sum h_{tm}^{(e-1)m}(u_m \cdot v_m)^2 d\sigma(u_m)$$

$$D(K)^{e-1} |e_{1+\varepsilon}(K) - s(K)|^2 \leq c_0 \left( \frac{1}{\omega_{2+\varepsilon}} \int \sum h_{K-s(K)}^{(1+\varepsilon)m} d\sigma - e_{1+\varepsilon}(K) \right)$$

Now the claim follows from the Blaschke-Santaló inequality. We have the following (see [1]).

Theorem 3.2. The following statements hold.

(1) Let $\varepsilon \geq 0$. If $e_{1+\varepsilon}(K) \leq 1 + \varepsilon$, then

$$\mathcal{H}(\tilde{K}, B)^2 \leq C(2 + \varepsilon)^5 \left( (1 + \varepsilon)^{\frac{1}{1+\varepsilon}} - 1 \right).$$

Here $C$ is a universal constant that does not depend on $(2 + \varepsilon)$.

(2) Let $1 + \varepsilon \in (-2 + \varepsilon, 0)$. If $e_{1+\varepsilon}(K) \geq 1 - \varepsilon$, then there exists an origin-centered ball of radius $(1 + \varepsilon), B_{1+\varepsilon}$, such that
\[ \delta_2 (\tilde{K} - e_{1+\varepsilon}(\tilde{K}), B_{1+\varepsilon}) \leq \left( 2c_2(D(\tilde{K}) + (1 + \varepsilon))^{3+\varepsilon} \varepsilon \right)^{1/2} + \left( c_0 D(\tilde{K})^{1-\varepsilon} \varepsilon \right)^{1/2} \]

Moreover, if \( \tilde{K} \) is origin-symmetric, then the last term on the right-hand-side can be dropped and \( D(\tilde{K}) \) can be replaced by \( \frac{1}{2} D(\tilde{K}) \). Here

\[ 1 \leq (1 + \varepsilon) \leq (1 - \varepsilon)^{1/2}, \ c_1 = \max \left\{ \frac{2 + \varepsilon}{(3 + 2\varepsilon)}, \frac{2 + \varepsilon}{1 + \varepsilon} \right\} \]

and \( c_0 \) is the constant from Lemma 3.1.

Proof. Case \( \varepsilon \geq 0 \): Since \( \varepsilon_{1+\varepsilon}(\tilde{K}) \leq 1 + \varepsilon \), we have

\[ \frac{1}{\omega_{2+\varepsilon}} \int \sum h_{\tilde{K}}^m d\sigma \leq \varepsilon_{1+\varepsilon}(\tilde{K})^{1/1+\varepsilon} \leq (1 + \varepsilon)^{1/1+\varepsilon} \]

The refinement of Urysohn's inequality in [10] completes the proof.

Case \( -(2 + \varepsilon) < 1 + \varepsilon < 0 \): Assume \( V(K) = V(B) \). Denote the support function of \( K - e_{1+\varepsilon}(K) \) by \( h_{1+\varepsilon}^m \) and the support function of \( K - s(K) \) by \( h_s^m \). Since \( s(K), e_{1+\varepsilon}(K) \) are in the interior of \( K \), both \( h_s^m \) and \( h_{1+\varepsilon}^m \) are positive functions.

Let us put

\[ f = \varepsilon_{1+\varepsilon}(K)^2 \]

By [20, Thm. 2.2], we have

\[ f^m = h_{\varepsilon}^{(1+\varepsilon)m}; \] 
\[ f = 1, \ (1 + \varepsilon)^2 = (2 + \varepsilon), (1 + 2\varepsilon) = \frac{2 + \varepsilon}{(3 + 2\varepsilon)}, \ c_1 = \max \{1 + \varepsilon, 1 + 2\varepsilon\} \]

Due to our assumption,

\[ \int \sum (h_{\varepsilon}^{1+\varepsilon m}) d\sigma \geq \int \sum h_{1+\varepsilon}^{(1+\varepsilon)m} d\sigma \geq \omega_{2+\varepsilon}(1 - \varepsilon). \]

By the Blaschke-Santaló inequality, we have

\[ \int \sum \frac{1}{h_{\varepsilon}^{(2+\varepsilon)m}} d\sigma \leq \omega_{2+\varepsilon} \]

(3.3)
From (3.2), (3.3), it follows that

$$1 - \varepsilon \leq \sum \frac{f h^{(1+\varepsilon)m}_s \, d\sigma}{\left( \int \frac{1}{h_{2+\varepsilon}^m} \, d\sigma \right)^{\frac{1}{2+\varepsilon}}}$$

(3.4)

$$(1 - \varepsilon)\omega_{2+\varepsilon} \leq \int \sum h^{(1+\varepsilon)m}_s \, d\sigma \leq \left( \int \sum \frac{1}{h_{2+\varepsilon}^m} \, d\sigma \right)^{\frac{1}{2+\varepsilon}} \omega_{2+\varepsilon}$$

Combining (3.1) and (3.4) we obtain

$$\sum \left| h^{(\frac{1}{2+\varepsilon})}_s - (1 + \varepsilon) \right|_{(K+\varepsilon)^2}^2 \leq c_1 \omega_{2+\varepsilon} D(K)^{2+\varepsilon} \varepsilon$$

(3.5)

where

$$(1 + \varepsilon)^{2+\varepsilon} := \omega_{2+\varepsilon} \left( \int \sum \frac{1}{h_{2+\varepsilon}^m} \, d\sigma \right)^{-1}, \quad 1 \leq (1 + \varepsilon) \leq (1 - \varepsilon)^{1+\varepsilon}$$

(3.6)

In view of (3.5) and (3.6) we have

$$\sum \left| h^m_s - (1 + \varepsilon) \right|_{(K+\varepsilon)^2}^2 \leq c_1 \omega_{2+\varepsilon} \left( D(K) + (1 + \varepsilon)^{\frac{1}{2}} \right)^2 D(K)^{2+\varepsilon} \varepsilon$$

(3.7)

If $K$ is origin-symmetric, then $\mathcal{g}(K) = \varepsilon_{1+\varepsilon}(K)$ and the proof is complete. Moreover, in this case we could have replaced $D(K)$ by $\frac{1}{2} D(K)$. Otherwise, to bound $\sum | h^m_{1+\varepsilon} - (1 + \varepsilon) |_{(K+\varepsilon)^2}$, note that by Lemma 3.1 we have

Therefore,

$$| \varepsilon_{1+\varepsilon}(K) - \mathcal{g}(K) |^2 \leq c_0 D(K)^{1+\varepsilon} \varepsilon$$

$$\sum \left| h^m_{1+\varepsilon} - (1 + \varepsilon) \right|_{(K+\varepsilon)^2}^2$$

$$\leq \sum \left| h^m_s - (1 + \varepsilon) \right|_{(K+\varepsilon)^2}^2 + \omega_{2+\varepsilon} \left| \varepsilon_{1+\varepsilon}(K) - \mathcal{g}(K) \right|$$

$$\leq \left( 2 c_1 \omega_{2+\varepsilon} (D(K) + (1 + \varepsilon)^{\frac{1}{2}}) \right)^2 + (c_0 \omega_{2+\varepsilon} D(K)^{2+\varepsilon} \varepsilon)^{\frac{1}{2}}.$$ 

**Remark 3.3.** The exponent $1/2$ in (1) is sharp; cf. [26]. Moreover, using [[27], Thm. [11],[8],[8]] it is also possible to give a stability result of order $1/(3 + \varepsilon)$ in (1) for the Hausdorff distance $d_{3\varepsilon}(K - \text{cent}(K), B)$; we leave out the details to the interested reader. By cutting off opposite caps of height $\varepsilon$ of the unit ball, one can see that the optimal order cannot be better than 1 in (2).

**Theorem 3.4.** Suppose $K$ is an origin-symmetric convex body with

$$\mathcal{E}_{-1}(K) \geq 1 - \varepsilon$$

for some $\varepsilon \in (0,1)$
Then there exists an origin-centered ball $B_{1+\varepsilon}$ of radius $1 \leq 1 + \varepsilon \leq (1 - \varepsilon)^{-1}$ such that

Moreover, we have

$$\delta_2(\hat{K}, B_{1+\varepsilon}) \leq D(\hat{K})\sqrt{\varepsilon}$$

$$\left(\frac{1}{2}D(\hat{K})\right)^{\frac{3}{2}} \leq \left(1 + \left(\frac{4\omega_{1+\varepsilon}}{\omega_{2+\varepsilon}}\right)^{\frac{3}{2}}\right)\frac{1}{1-\varepsilon}$$

Proof. Set $h^m = h^m_{\hat{K}}$. We have

$$\sum \frac{1}{h^m} \frac{1}{h^m d\sigma} = 1 - \frac{1}{2} \sum \left( \frac{1}{h^m} \frac{1}{h^m d\sigma} \right)^{\frac{3}{2}} \omega_{2+\varepsilon}^{\frac{1}{2}} \omega_{2+\varepsilon}^{\frac{1}{2}} \left(\hat{K}_x\right)^{\frac{3}{2}}$$

By our assumption and the Blaschke-Santaló inequality,

Therefore,

$$\int \sum \frac{1}{h^m} d\sigma \geq \omega_{2+\varepsilon}(1 - \varepsilon), \sum \frac{1}{h^m} d\sigma \leq \omega_{2+\varepsilon}$$

Combining these inequalities, we obtain

$$1 - \varepsilon \leq \sum \frac{1}{h^m} \frac{1}{h^m d\sigma} \left(\frac{1}{h^m d\sigma}\right)^{\frac{3}{2}} \omega_{2+\varepsilon}^{\frac{1}{2}} \omega_{2+\varepsilon}^{\frac{1}{2}} \left(\hat{K}_x\right)^{\frac{3}{2}}$$

$$\leq \left(\sum \frac{1}{h^m} d\sigma\right)^{\frac{3}{2}} \omega_{2+\varepsilon}^{\frac{1}{2}} \omega_{2+\varepsilon}^{\frac{1}{2}} \left(\hat{K}_x\right)^{\frac{3}{2}} \omega_{2+\varepsilon}^{\frac{1}{2}} \omega_{2+\varepsilon}^{\frac{1}{2}} \left(\hat{K}_x\right)^{\frac{3}{2}}$$

$$\sum |h^m - (1 + \varepsilon)|^2 \leq \omega_{2+\varepsilon} D(\hat{K}) \varepsilon$$

where $(1 + \varepsilon)^2 = \omega_{2+\varepsilon} \left(\sum h^m d\sigma\right)^{-1}$ and $1 \leq 1 + \varepsilon \leq (1 - \varepsilon)^{-1}$.

Next we estimate the diameter from above. Define

$$S = \left\{ v_m \in S^{1+\varepsilon}; \sum \frac{1}{h^m_{\hat{K}}(v_m)} \leq \frac{R^2}{\varepsilon} \right\}$$
where $R := \max \hat{h}_k^{\|u_m\|} \leq \hat{h}_k^{\|u_m\|}$ for some vector $u_m \in S^{1+\varepsilon}$. We may assume $R > 1$. Then by the

$$
(1-\varepsilon)\omega_2^{1+\varepsilon} \leq \int_S \sum \frac{h_k^{\|u_m\|}}{h_k} d\sigma + \int_{\sigma^c} \sum \frac{1}{h_k^{\|u_m\|}} d\sigma \leq \sum \left( \int_S \frac{1}{h_k^{\|u_m\|}} d\sigma \right)^{\frac{1}{2}} |S|^\frac{1}{2} + \frac{|\sigma^c|}{R^2}
$$

Blaschke-Santaló inequality we have

$$
\leq (\omega_2^{1+\varepsilon})^{\frac{1}{2}} |S|^\frac{1}{2} + \frac{\omega_2^{1+\varepsilon}}{R^2}
$$

Moreover, by convexity we have $\sum h_k^{\|u_m\|} \geq \sum R |u_m \cdot v_m|$ for all $v_m \in S^{1+\varepsilon}$. Hence if $v_m \in S$, then $\sum |u_m \cdot v_m| \leq R^{\frac{1}{1+\varepsilon}}$. Now using $\frac{\pi}{2} - \arccos x \leq 2x$, $\forall x \in [0,1]$ we obtain

Therefore,

$$
\frac{1}{2} |S| \leq \omega_2^{1+\varepsilon} \int_{\arccos R^{\frac{1}{1+\varepsilon}}}^{\frac{\pi}{2}} \sin^{1+\varepsilon} \theta d\theta \leq \frac{2\omega_2^{1+\varepsilon}}{R^2}
$$

We give the proofs of the main theorems (see [27]).

$$
1 - \varepsilon \leq \left( 1 + \left( \frac{4\omega_2^{1+\varepsilon}}{\omega_2^{1+\varepsilon}} \right) \right) \frac{1}{R^2}
$$

**Proof of Theorem 1.1.** Suppose $m_0 \leq h_K^{m(e)} dS_K / d\sigma \leq M$. Therefore by the $L_{1+\varepsilon}^+$- Minkowski inequality,

$$
\frac{m_0}{2 + \varepsilon} \int B(1+\varepsilon) \frac{1}{V(B) \bar{z}^{1+\varepsilon} V(K) \bar{z}^{1+\varepsilon}} \leq \frac{1}{2 + \varepsilon} \sum \int h_K^{1+\varepsilon} h_K^{m(-\varepsilon)} dS_K
$$

$$
= \frac{V(K) \bar{z}^{1+\varepsilon}}{V(B) \bar{z}^{1+\varepsilon}} \leq M
$$

$$
\leq \sum \frac{V(B) \bar{z}^{1+\varepsilon}}{V(B) \bar{z}^{1+\varepsilon}} \int h_K^{m(-\varepsilon)} dS_K \leq M
$$

Hence $\mathcal{E}_{1+\varepsilon}(\tilde{K}) \leq \mathcal{R}_{1+\varepsilon}(\tilde{K})$, and by Theorem 3.2 the proof is complete.

**Proof of Theorem 1.2.** Assume $m_0 \leq h_{1+\varepsilon}^{m(e)} dS_K / d\sigma \leq M$. Then by the $(L)_{2+\varepsilon}$- Minkowski inequality for $\varepsilon \geq 0$ we have

Therefore,
\[
\frac{1}{2 + \epsilon} \int \sum \frac{1}{h_K^{m(2\epsilon+1)}} h_K^{-m(-\epsilon)} dS_K \geq V(B)
\]

Owing to (2.4) for \( \epsilon \geq 0 \) we have

\[
\frac{M}{2 + \epsilon} V(K)^{\frac{2\epsilon+1}{2\epsilon}} \int \sum \frac{1}{h_K^{m(2\epsilon+1)}} d\sigma \geq V(K)^{\frac{1}{2\epsilon}} V(B).
\]

(3.8)

\[
V(K) \geq \frac{m_0}{2 + \epsilon} \int \sum h_K^{(1+\epsilon)m} d\sigma \geq m_0 V(K)^{\frac{1+\epsilon}{2\epsilon}} V(B)^{\frac{1}{2\epsilon}}
\]

and hence for \( \epsilon \geq -1 \),

\[
V(K)^{\frac{1}{2\epsilon}} \geq m_0 V(B)^{\frac{1}{2\epsilon}}
\]

(3.9)

Since \( e_{-1}(K) = 0 \), in view of (3.8) we obtain \( e_{-1}(K) \geq R_{1+\epsilon}(K)^{-1} \). The claim follows from Theorem 3.4.

**Remark 3.5.** It is clear from the proofs of Theorem 1.1 and Theorem 1.2, that if \( K \) has only a positive continuous curvature function, then the same conclusions hold.

Remark 3.6. Applying the Blaschke-Santaló inequality to the left-hand side of (3.8), we obtain

This combined with (3.9) yields

\[
\left( \frac{V(K)}{V(B)} \right)^{\frac{2\epsilon + 1}{2\epsilon}} \leq M
\]

\[
m_0 \leq \left( \frac{V(K)}{V(B)} \right)^{\frac{2\epsilon + 1}{2\epsilon}} \leq M
\]

Hence in the class of origin-symmetric bodies if \( V(K) = V(B) \), then for any \( \epsilon \geq -1 \) the \( (K + \epsilon)_{1+\epsilon} \) curvature function attains the value 1 at some point; see also Question 3.

**Proof of Theorem 1.3.** Define \( \check{\varepsilon}_{1+\epsilon} : \mathbb{P}^{2+\epsilon}_0 \rightarrow (0, \omega) \) by

\[
\check{\varepsilon}_{1+\epsilon}(h_K^{m}) = \left( \int \sum h_K^{1+\epsilon m} d\sigma \right)^{\frac{2\epsilon}{1+\epsilon}} / V(K + \epsilon)
\]

By the divergence theorem we have

\[
\sum (\text{grad} \check{\varepsilon}_{1+\epsilon})(h_K^{m}) = \sum \frac{h_K^{(1+\epsilon) m} \left( \int h_K^{1+\epsilon m} d\sigma \right)^{\frac{2\epsilon}{1+\epsilon}}}{V(K)^2} \left( 2 + \epsilon \right) \frac{V(K)}{h_K^{1+\epsilon m} d\sigma} - h_K^{(1-\epsilon) m} f_{1+\epsilon}^m
\]
By [25], Sec. 3.13 (ii) and [25, p. 80], there exist $c_2, \delta > 0$, such that for any $K$ with $\sum |h_K^m - 1| \leq \delta$, there holds

$$\left| \xi_{1+\varepsilon}(K) - \xi_{1+\varepsilon}(B) \right|^2 \leq c_2 \sum |(\text{grad } \xi_{1+\varepsilon})(h_K^m)|^{(K-\varepsilon)}$$

Assuming $m_0 \leq h_K^{(1+\varepsilon)m} H_K^m \leq M$ gives

$$m_0 \leq \frac{(2 + \varepsilon) V(K)}{\int \sum h_K^{(1+\varepsilon)m} d\sigma} \leq M$$

This in turn implies $|\xi_{1+\varepsilon}(K)|^{2+\varepsilon} - 1| \leq c_3 (R_{1+\varepsilon}(K) - 1)^2$, as well as

$$\xi_{1+\varepsilon}(K) \geq \left(1 + c_3 (R_{1+\varepsilon}(K) - 1)^2 \right)^{1/2+\varepsilon}$$

Due to Theorem 3.2, the proof is complete.

**Proof of Theorem 1.5.** Suppose $m_0 \leq H_K \leq M$. By [3, Lem. [18],

$$V(K) \geq \frac{\pi}{\sqrt{M}}$$

(3.10)

In fact, the lemma states that if $V(K) = \pi$, then centro-affine curvature at some point attains 1. Therefore, since $V(\sqrt{\pi/V(K)} K) = \pi$, the function $(V(K)/\pi)^2 H_K$ takes the value 1 at some point. Hence using (3.10) and the Hölder inequality we obtain

$$V(K)V(K^*) \geq \sum \left( \int \frac{h_K^{m} f_K^m H_K^*}{4 \int h_K^{m} f_K^m d\sigma} \right)^3 \geq m_0 V(K)^2 \geq \pi^2 \frac{m_0}{M}$$

If the Santaló point is at the origin, then we can obtain a slightly better lower bound for the volume product. By [28], we have

$$\sum H_K(u_m, H_K^*(u_m)) = 1$$

where $u_m$ and $u_m^*$ are related by $\Sigma(v_K^{-1}(u_m), v_K^{-1}(u_m^*)) = 1$. Since $K^* = K^*$, this yields

$$\frac{1}{M} \leq H_{K^*} \leq \frac{1}{m_0}, V(K^*) \geq \pi \frac{m_0}{\sqrt{M}}$$

Therefore, $V(K)V(K^*) \geq \pi^2 \frac{m_0}{\sqrt{M}}$. Now in both cases, the result follows from [29]. The third claim is exactly [29, Cor. [9].
Question 3. Given the previous argument, we would like to raise a question. Suppose $K \in \mathcal{F}_0^{2+\epsilon}, \epsilon \geq 0$, and $V(K) = V(B)$. Is it true that the centro-affine curvature of $K$ attains the value 1 at some point?

**Proof of Theorem 1.6.** For all $\ell \in G(K + \epsilon)(2 + \epsilon)$, we have

$$s(\ell K) = \ell s(K) = 0, \quad d_{B_{2+\epsilon}}(\ell K, B) = d_{B_{2+\epsilon}}(K, B).$$

Thus we may assume without loss of generality that for some $\delta > 0$ to be determined.

$$\sum |h_K^m - 1|_{c^2} \leq \delta$$

Define the functional $\mathcal{P}: \mathcal{P}_0^{2+\epsilon} \rightarrow (0, \infty)$ by

We have

$$\mathcal{P}(K + \epsilon) = \mathcal{P}(h_K^m) = \frac{1}{V(K + \epsilon)V((K + \epsilon)^*)}$$

$$\sum (\text{grad } \mathcal{P})(h_K^m) = \sum \mathcal{P}^2(K) \left( \frac{V(K)}{h_K^m} - \frac{V(K^*)}{h_{K^*}^m} \right)$$

$$= \sum \frac{V(K^*)}{h_{K^*}^m} \mathcal{P}^2(K) \left( \frac{V(K)}{h_K^m} - \frac{1}{h_K} \right)$$

(3.11)

By [25], Sec. 3.13 (ii), there exist $\delta, c_2 > 0$ and $\alpha \in (0, 1/2]$, such that for any $K$ with $\sum |h_K^m - 1|_{c^2} \leq \delta$, we have

$$\left| \frac{1}{V(K)V(K^*)} - \frac{1}{V(B)^2} \right|^{1-\alpha} \leq c_2 \sum |(\text{grad } \mathcal{P})(h_K^m)|_{(K + \epsilon)^*}^2$$

(3.12)

By [25], p. 80] and[ 30], Lem. 4.1, 4.2] we can choose $\alpha = 1/2$

We estimate the right-hand side of (3.12). Note that $m_0 \leq H_K \leq M$ implies that

Therefore we obtain

$$\frac{1}{M} \leq \frac{V(K)}{V(K^*)} = \sum \int \frac{h_K^m f_K^m}{h_K^m f_K^m H_K^m} d\sigma \leq \frac{1}{m_0}$$

$$\frac{1}{M} \leq \frac{V(K)}{V(K^*)} \leq \frac{1}{m_0} \text{ and } \left| \frac{V(K^*)}{V(K^*)} - \frac{1}{H_{K^*}} \right| \leq \frac{M - m_2}{M m_0}$$

(3.13)

On the other hand, by (3.13) and the Blaschke-Santaló inequality,

$$V(K^*)^2 \leq MV(B)^2$$

(3.14)
Putting (3.11), (3.12), (3.13), and (3.14) all together we arrive at

\[
\left| \frac{1}{V(K)V(K')} - \frac{1}{V(B)^2} \right| \leq c_3 (R_{(2+\epsilon)}(K) - 1) \sum \frac{p^2(K)\left| h_K^{-m(1+\epsilon)} \right|_{(K')}^2}{V(K')} \]

Since we are in a small neighborhood of the unit ball, the term

\[
\sum \frac{p^2(K)\left| h_K^{-m(1+\epsilon)} \right|_{(K')}^2}{V(K')} \]

is bounded. Using again the Blaschke-Santaló inequality we obtain

\[1 - c_4 (R_{(2+\epsilon)}(K) - 1)^2 \leq \frac{V(K)V(K')}{V(B)^2}\]

In view of [ [19], Thm. 1.1], the proof is complete.

References