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Numerical technique of Dynamical models with Applications to Chaotic Systems

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Abstract: *In this article a new numerical scheme for dynamical models with applications to chaotic systems was developed, we present the analysis of errors in general. The novel numerical method was used to address both linear and nonlinear fractional dynamical systems. The technique was used two (linear and nonlinear) systems to be solved of the fractional ordinary differential equations as well as fours nonlinear chaotic models.*

Keywords: *Fractional differential system, AB operators, Stability, chaotic.*

1. Introduction

Differential equations play both active roles in the description of natural and complicated events in fluid mechanics, biology, and applied physics, fractional equations it is a powerful tool for describing living phenomena in physics and engineering [1-11]. Therefore, numerical scheme has been that has gained acceptance as a useful system. Therefore, numerical scheme has been recognized as an effective tool for solving systems. We dedicate this article the creation of a novel numerical approach that incorporates the fundamental theorem of fractions and the two-stage Lagrangian polynomial [12]. The numerical approach will be used to resolve both linear and nonlinear systems of ordinary fractional equations.

Several methodologies, including physics and engineering, have been applied to problems in management, economics and biology, among others [13-18], recent attention has been drawn to the fractional SIR Model due to the spread of diseases such as Covid-19. Furthermore, we will apply this technique to several chaotic model [19-21], The numerical simulation results shown in both situations comprise and three-dimensional phase pictures with varying parameters. The novelty of this study is that it provides a numerical solution for fractional derivative order in linear and nonlinear fractional dynamical systems.

The goal of this research was to lay the groundwork for using fractional systems in other fields of science and technology. This method's value resides in the fact that it can be applied to a wide variety of models from disease models and chaos models to additional models in pathology and dynamical models in order to locate a numerical solution.

2. Preliminaries

Definition 1 Riemann-Liouville fractional integral operator of order $\alpha > 0$ for a function $y(\tau)$ is given by [22]:

$$D^{\alpha}y(t) := \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}y^n(\tau)d\tau. = I^{n-\alpha}y^n(t), t > 0 \qquad (1)$$

Definition 2 The fractional integral of order α of a function f is defined as [23]

$$I_t^{\gamma}y(t) := \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}y(t) + \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\int_0^t y(\tau)d\tau. t \geq 0, 0 < \alpha < 1 \qquad (2)$$

Definition 3 The Mittag-Leffler function is a generalization of the exponential function. This function can be expressed as follows:

$$E_{\alpha}(t) = \sum_{k=0}^{\infty}\frac{t^k}{\Gamma(\alpha k+1)} \qquad (3)$$

Definition 4 For $y\epsilon H^1(0,T)$, $T > 0, \alpha\epsilon(0,1]$ Then the AB fractional operator [24] y(t) in the Riemann–Liouville is given by

$$ {}^{AB}_{0}D_t^{\alpha}y(t) := \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int_0^t y(\tau)E_{\alpha}\left(\frac{\alpha}{1-\alpha}(t-\tau)^{\alpha}\right)d\tau.\ 0 < \alpha < 1 \qquad (4)$$

In this expression $B(\alpha)$ satisfies the condition $B(0) = B(1) = 1$.

Definition 5 For y ∈ H^1 (0, T), T > 0Then the AB fractional operator [24] y(t) in the Caputo sense is given by

$$ {}^{AB}_{0}D_t^{\alpha}y(t) := \frac{B(\alpha)}{1-\alpha}\int_0^t \frac{dy(\tau)}{d\tau}E_{\alpha}\left(\frac{\alpha}{1-\alpha}(t-\tau)^{\alpha}\right)d\tau.\ 0 < \alpha < 1 \qquad (5)$$

In this expression $B(\alpha)$ satisfies the condition $B(0) = B(1) = 1$.

3. Applications:

3.1 The new numerical scheme

$$x_{n+1} = x_0 + \frac{(1-\alpha)}{ABC(\alpha)} f\big(t_n, x(t_n)\big) + \frac{\alpha}{ABC(\alpha)} \sum_{k=0}^{n} \Bigg(\frac{h^\alpha f(t_k, x_k)}{\Gamma(\alpha+2)} \big((n-k+1)^\alpha (n-k+2+\alpha) - (n-k)^\alpha (n-k+2+2\alpha)\big) - \frac{h^\alpha f(t_{k-1}, x_{k-1})}{\Gamma(\alpha+2)} \big((n-k+1)^{\alpha+1} - (n-k)^\alpha (n-k+1+\alpha)\big) \Bigg)$$

$$y_{n+1} = y_0 + \frac{(1-\alpha)}{ABC(\alpha)} f\big(t_n, y(t_n)\big) + \frac{\alpha}{ABC(\alpha)} \sum_{k=0}^{n} \Bigg(\frac{h^\alpha f\big(t_k, y_k\big)}{\Gamma(\alpha+2)} \big((n-k+1)^\alpha (n-k+2+\alpha) - (n-k)^\alpha (n-k+2+2\alpha)\big) - \frac{h^\alpha f\big(t_{k-1}, y_{k-1}\big)}{\Gamma(\alpha+2)} \big((n-k+1)^{\alpha+1} - (n-k)^\alpha (n-k+1+\alpha)\big) \Bigg)$$

$$z_{n+1} = y_0 + \frac{(1-\alpha)}{ABC(\alpha)} f\big(t_n, z(t_n)\big) + \frac{\alpha}{ABC(\alpha)} \sum_{k=0}^{n} \Bigg(\frac{h^\alpha f(t_k, z_k)}{\Gamma(\alpha+2)} \big((n+1-k)^\alpha (n-k+2+\alpha) - (n-k)^\alpha (n-k+2+2\alpha)\big) - \frac{h^\alpha f(t_{k-1}, z_{k-1})}{\Gamma(\alpha+2)} \big((n-k+1)^{\alpha+1} - (n-k)^\alpha (n-k+1+\alpha)\big) \Bigg)$$

The use of the novel numerical approach [16] for solving fractional differential equations is covered in the next section.

3.2. linear systems of the fractional ordinary differential equations:

$$D_t^\lambda S = aS_1 - bS_2 \qquad (6)$$

$$D_t^\lambda S = cS_1 - dS_2$$

We apple AB operators in system (6-7) we get:

$${}_{0}^{AB}D_t^\lambda S = aS_1 - bS_2$$

$${}_{0}^{AB}D_t^\lambda S = cS_1 - dS_2 \qquad (7)$$

Where a=2, b=1, c=4, d=3.

We simulate the numerical solution using the specified numerical approach., the estimation of error is given

Table 1: Estimation of Error

$0 < \alpha_i \le 1$	$h = \frac{1}{\Delta t}$	$\lVert R_\alpha \rVert = \left\lVert y_{app(t)} - y_{exac(t)} \right\rVert$
α_1 **= 0.94**	10	0.4875
α_2 **= 0.97**	100	0.033
α_3 **=0.99**	1000	0.0057918680

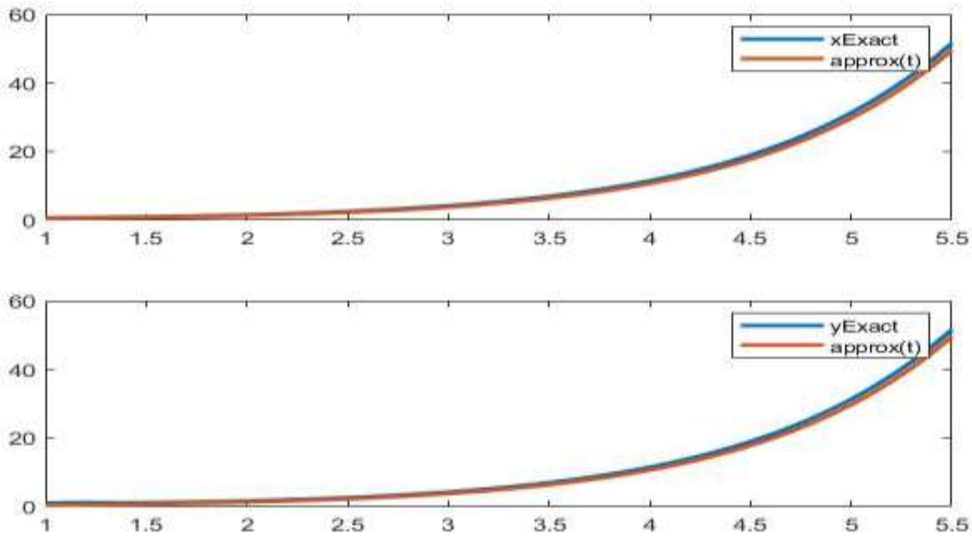

Fig.1 Comparison of exact solution and numerical solution for h = 0.1.

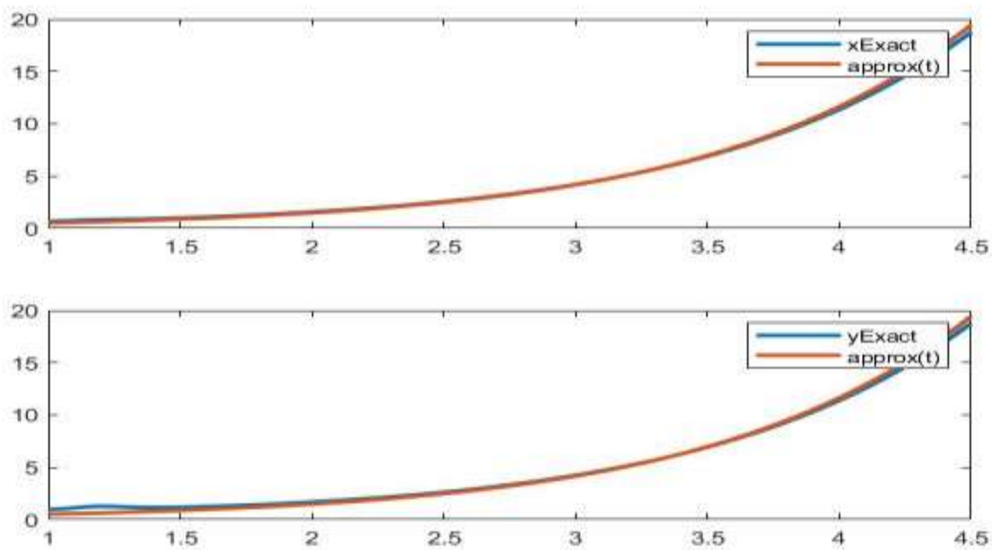

Fig.2 Comparison of numerical solution and exact solution for h = 0.01.

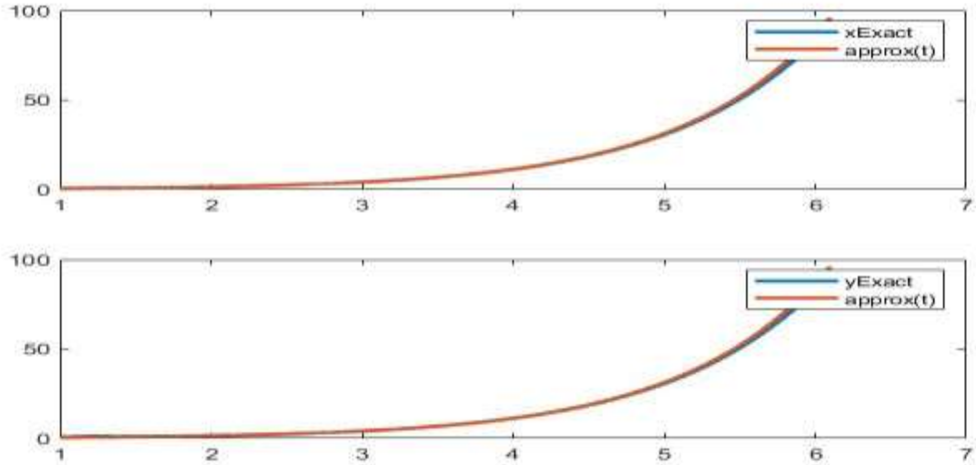

Fig.3 Comparison of numerical solution and exact solution for h = 0.001.

3.3. Nonlinear systems of the fractional ordinary differential equations:

$$\dot{S_1} = \theta(S_2 + S_1)$$
$$\dot{S_2} = -S_2 - S_1 S_3 + \beta S_1$$
$$\dot{S_3} = -S_3 + S_1 S_2 \qquad (10)$$

We apple AB operators in system (6-7) we get:

$$^{AB}_{0}D^{\lambda}_{t} S_1 = \theta(S_2 + S_1)$$
$$^{AB}_{0}D^{\lambda}_{t} S_2 = -S_2 - S_1 S_3 + \beta S_1 \qquad (11)$$
$$^{AB}_{0}D^{\lambda}_{t} S_3 = -S_3 + S_1 S_2$$

Where $0 < \lambda \leq 1$.

Equilibria: The three equilibrium points obtained by solving (8) are $E_1 = (\sqrt{\beta - 1}, \sqrt{\beta - 1}, \beta - 1)$, $E_2 = (0,0,0)$, $E_3 = (-\sqrt{\beta - 1}, -\sqrt{\beta - 1}, \beta - 1)$

Jacobean matrix of the system (8) is given

$$J(E_1) = \begin{bmatrix} -\theta & \theta & 0 \\ \beta - S_3 & -1 & -S_1 \\ S_2 & S_1 & -1 \end{bmatrix} \qquad (12)$$

At the parameter values $\theta = 14, \beta = 110$,the equilibrium points are

$E_1 = (10.45\,, 10.45\,, 109)$,(13) $E_3 = (10.45\,, 10.45\,, 109)$, $E_2 = (0,0,0)$

The Jacobean matrix at $(10.45\,, 10.45\,, 109)$

$$J(E_1) = \begin{bmatrix} -15 & 15 & 0 \\ 1 & -1 & -10.45 \\ 10.45 & 10.45 & -1 \end{bmatrix} \qquad (14)$$

The characteristic equation obtained from (6) at equilibrium E_i (S_1^*, S_2^*, S_3^*) is as in the following equation.

$\omega^3 + 17\omega^2 + 125\,\omega$+(3270=0 (14)

The solution of the characteristic equation gives the eigenvalues. similarly, the eigenvalues corresponding to E_2 and E_3 are calculated.

Table 2: Equilibria of (11) and their eigenvalues

E_i	Equilibria	Eigenvalues	Stability
E_1	$(10.45, 10.45, 109)$	(1.1507±12.96i, - 19.31)	Unstable
E_2	$(0,0,0)$	(33.2189, -1, -49.22)	Unstable
E_3	$(-10.45, -10.45, 109)$	(1.1507±12.96i, -19.31)	Unstable

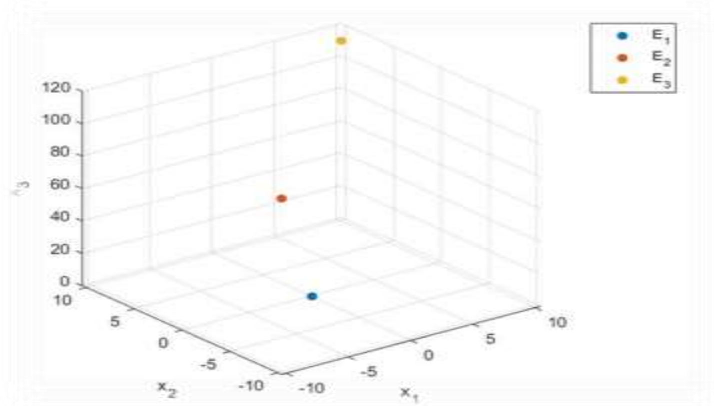

Fig.4. Distribution equilibrium points of the fractional-order system.

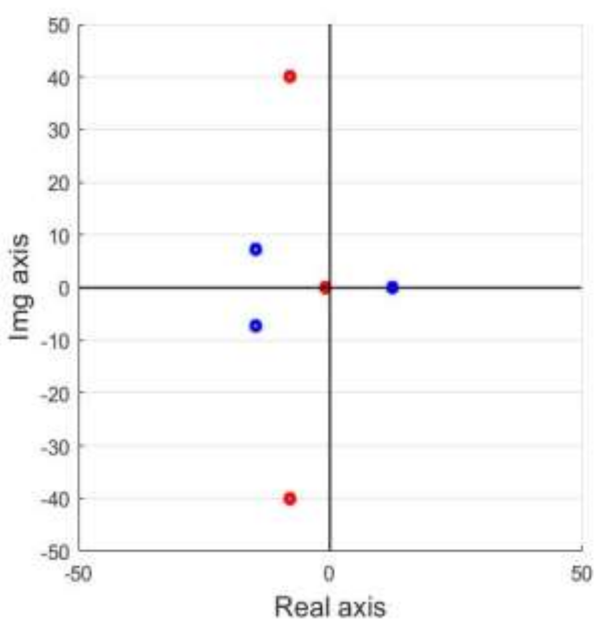

Fig.5. Stability of the fractional-order system.

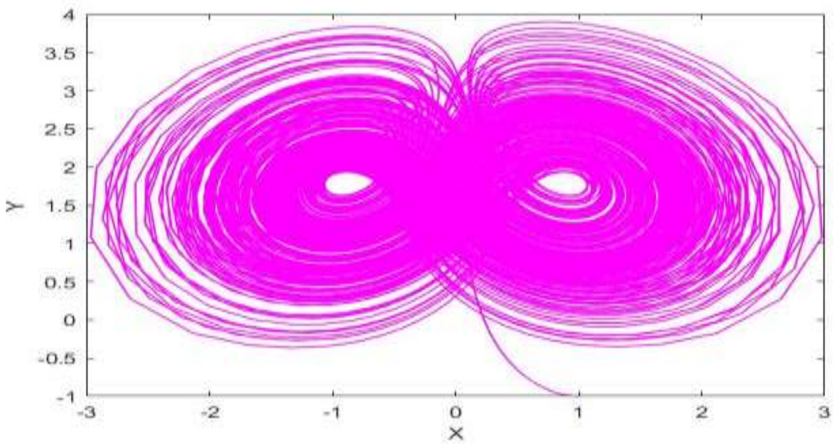

Fig.6. Chaotic attractor for α = 0.98, h=0.1and t=2000 in a y-x plane.

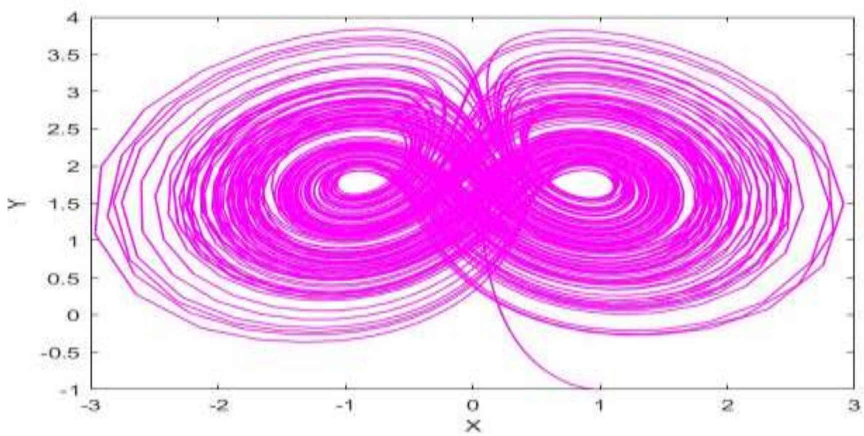

Fig.7. Chaotic attractor for α = 0.98, h=0.1and t=1500 in a y-x plane.

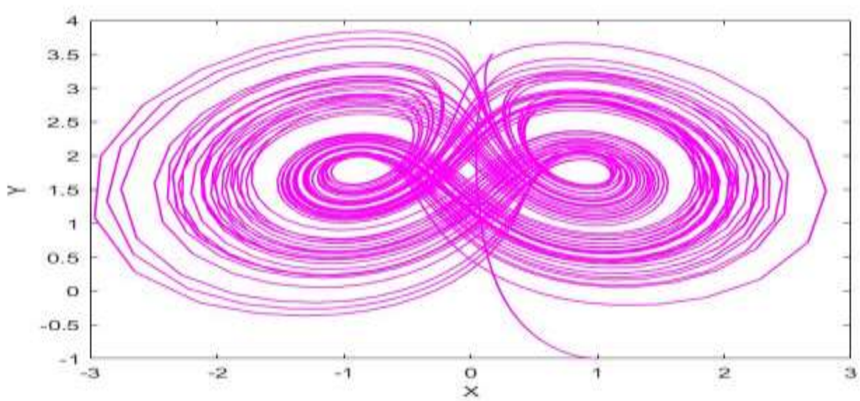

Fig.8. Chaotic attractor for α = 0.98, h=0.1and t=500 in a y-x plane.

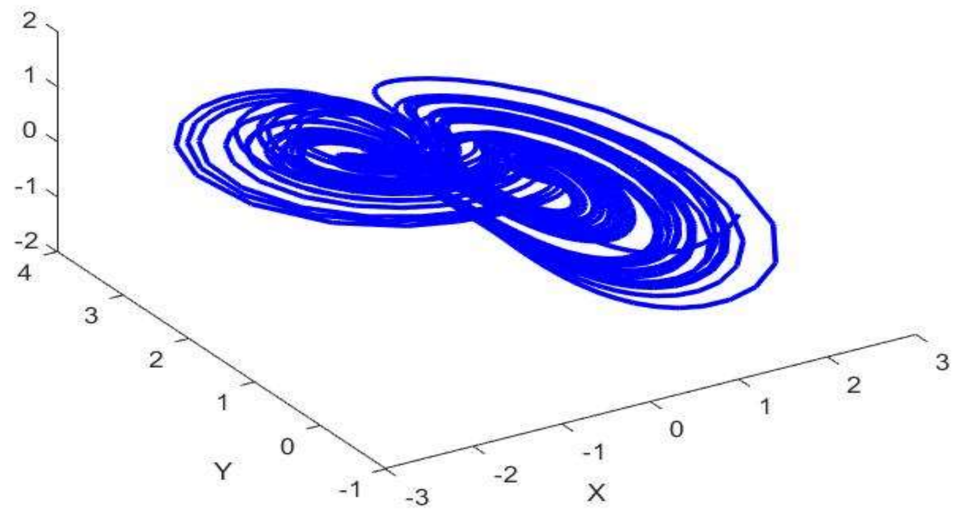

Fig.9. Chaotic attractor for α = 0.98, h=0.1and t=500 in a y-x-z space.

4. **Conclusion:**

We proposed a new numerical approach to solve fractional differential equations (Linear and nonlinear) made up from this kind of derivative. The novel scheme technique combines polynomials and Lagrange theorems with fractional calculus' basic theorem. When compared to precise answers, the suggested technique can be demonstrated to be extremely accurate. The method is simple, effective and can be widely applied as it shows chaos, the stability analysis of this method has discussed with error estimation. Effective and adaptable to many fractional systems, this technique is widely used.

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