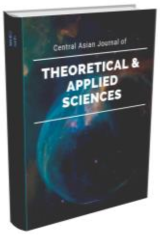

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Article

On Cubic Ideals of GS-Algerasb

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Abstract: This study explores the use of Nearpod, a dynamic e-learning platform, to enhance students' English-speaking skills within an Intensive Speaking class at Makassar State University. The primary objective is to assess the platform's effectiveness in improving student engagement and participation in online language learning. Employing a class action research design, the study involved 46 students, divided into experimental and control groups, over two cycles. Data were collected using observation checklists, pre-tests, post-tests, and student surveys. The results indicated a 22.08% increase in student activeness from Cycle 1 to Cycle 2, moving participants from the "adequate" to the "good" category. The experimental group, using Nearpod, performed marginally better than the control group, with low-performing students showing significant improvement. Surveys revealed that students found the platform engaging, comfortable, and effective for enhancing learning focus and interaction. These findings underscore Nearpod's potential as an innovative tool for fostering active participation and improving English-speaking skills in online learning environments.

Keywords: Cubic of GS-Algebra,Cubic Ideal of GS-Algebra,Cubic GS-Algebra Homomorphisim

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1. Introduction

BCK-algebras form an important class of logical algebras introduced by Ise´ki and were extensively investigated by sev- eral researchers. The class of all BCK-algebras isquasivariety. Ise´kiandTanakaintroducedtwoclassesofabstractalgebras, BCK-algebras and BCI-algebras [1–3]. In connection with this problem, Komori [4] introduced a notion of BCC- algebras.

Ravi Kumar Bandru and N. Ra_ [12] introduce a new notion called a G-algebra, which is a generalization of QS-algebra [14]. The concept of 0- commutative, G-part and medial of a G-algebra are introduced and studied their properties. In 1998, Y. B. Jun et al. [16] introduced a new class of algebras related to BCI-algebras and semigroups, called a BCI-semigroup/BCI-monoid/BCI-group. In 1998, for the convenience of study, Y. B. Jun et al. [12] renamed the BCI-semigroup (resp, BCI- monoid and BCI-group) as the IS-algebra (resp. IM-algebra and IG-algebra) and studied further properties of these algebras (see [5] and [13]). In [6], E. H. Roh et al. introduced the concept of a p&I-ideal in an IS-algebra, and gave necessary and suffcient conditions for an I-ideal to be a p&I-ideal, and also stated a characterization of PS-algebras by p&I-ideals. Jun et al. [16] introduced the notion of cubic subalge- bras/ideals in BCK/BCI-algebras,and then they investigated several properties.

In this paper, we introduce the notion of cubic ideals of GS-algebras and then we study the homomorphic image and inverse image of cubic GS-ideals.

2. Materials and Methods

This study investigates the use of Nearpod as an innovative teaching tool to enhance English-speaking skills among students in an English Language Education program. The methodology is based on class action research and includes the following components:

Participants:

The study involved 46 students enrolled in an Intensive Speaking class at Makassar State University. The participants were randomly selected and divided into experimental and control groups.

Study Design:

The research was conducted over two cycles (Cycle 1 and Cycle 2), each consisting of three meetings. Both qualitative and quantitative methods were employed, including observation and surveys.

Instruments:

a. Observation Checklist: Used to assess students' active participation, categorized into four levels: Very Active, Active, Less Active, and Not Active.
b. Questionnaire: Designed to capture students' responses, engagement, and comfort level with Nearpod during online learning.
c. Pre-test and Post-test: Administered to measure improvement in speaking skills.

Implementation:

a. Teachers integrated Nearpod into online sessions using features such as quizzes, polls, videos, and open-ended questions to create an interactive environment.
b. Students worked in groups and individually to complete projects and engage with course material.

Data Analysis:

a. Observation data were analyzed using descriptive statistics to compare levels of student activeness across cycles.
b. Survey responses were summarized to gauge student perception of Nearpod.
c. Performance data were analyzed to evaluate differences between experimental and control groups, with a focus on low-performing students.

3. Results and Discussion

Preliminaries

In this section we recall some preliminary definitions and results to be usedin the sequel.

Definition 2.1. ([7]) Analgebra (X ; *,0) is called aG-algebra if it satisfies the following axioms

$$G_1)\ x * x = 0,$$

$$(G_2)\ \big(x * (x * y)\big) = y. for\ all\ x, y \in X$$

Proposition2.2.

([13])In aG-algebr aX,the following properties are true:

a. x * x = x,
b. 0 * (0 * x) = x,
c. (x * (x * y) * y = 0,
d. x * y = 0 implies x = y

e. 0 * x = 0 * y implies x = y for all x , y ∈ X

Definition2.3. ([7])

Let X be a G-algebra.Forany subset S of X, we define G(S) ={x*S: 0 * x*x }. In particular, If S = X then we saythat G(X) is the G- Part of a G- algebra

Definition2.4.([7]

For any G-algebraX, the set B(X) = {x ∈ X: 0 *x = 0} is called ap-radecl of X. If B(x) = {0}, a G-algebra X is said to be p-semisimple.

The following property is obvious G(X)∩B(X)={0}.

Theorem2.5.([7]).

Let (X;,0)be a G-algebra.IfG(X)=X,then X is p-semisimple.*

Definition 2.6.([7])

Anonempty subset S of aG-algebraX is called asubalgebra of X ifx * y∈S , for allx, y ∈S

Proposition2.7.([7])

Let(X;*,0)beaG-algebra.Then x∈G(X)if *and only* if 0*x∈G(X).

Theorem 2.8.([7})

Let (X;*,0)be a G-algebra.If Sis a subalgebra of X,then G(X)∩S=G(S).

Theorem2.9.([7])

Let X be aG-algebra.If G(X)=X,then X is p-semisimple.

Theorem2.10.([7})

EveryG-algebrasatisfying(x*y) *(x*z)= z*y is a BCI-algebra.

Definition2.11.([10])

AnIS-algebraisanonemptysetXwithtwobinaryoperation

"*","."And constant 0 satisfying the axioms:

$I(X) * (X,* ,0) isa\ BCI - algebra,$

$S(X)\ *\ (X,.) isasemigroup,$

$x.(y * z)\ *\ (x.y)\ *\ (x.z) and (x * y).z *\ (x.z)\ *\ (y.z). Forallx, y, z \in X$

GS–Algebras

a. Cubic ideal of GS-Algebras

We recall that a cubic set ℵ in a set ψ is the structure ψ={(ϱ,$\tilde{\alpha}$_ψ(ϱ),ϑ_ψ(ϱ):ϱ∈ℵ}, where

$\tilde{\alpha}$_ψ:ℵ →D [0,1]suchthat $\tilde{\alpha}$_ψ(ϱ)=[ϑ_ψ^L (ϱ),ϑ_ψ^U (ϱ)] is an interval valued fuzzy set inℵ and ϑ_ψis afuzzy set in ℵ.We write acubic set by as follows.

ψ = ($\tilde{\alpha}$_ψ,ϑ_ψ) and we can define the level subset of ψ =($\tilde{\alpha}$_ψ,ϑ_ψ)which is denoted by U(ψ,ś,t) as follows U(ψ,ś,t) ={ϱ∈ℵ:$\tilde{\alpha}$_ψ(ϱ) ≥ś,ϑ_ψ≤t},for every [0,0] ≤ ś ≤[1,1]and t∈[0,1].

Definition 4.1: Let ℵ

is said to be a cubic ideal

$(s_1)\ \tilde{\alpha}_\psi(0) \geq \tilde{\alpha}_\psi(\rho) and\ \vartheta_\psi(0) \leq \vartheta_\psi(\rho),$

$\vartheta_\psi(y * z)\}\quad (s_2)\tilde{\alpha}_\psi(x * z) \geq rmin\{\tilde{\alpha}_\psi(x * y), \tilde{\alpha}_\psi(y * z)\} and\ \vartheta_\psi(\rho * z) \leq max\ \{\vartheta_\psi(\rho * y),$

$for\ all\ \rho, y, z \in$ **ℵ.**

$\vartheta_\psi(y.z)\}\quad (s_3)\ \tilde{\alpha}_\psi(x.z) \geq rmin\{\tilde{\alpha}_\psi(x.y), \tilde{\alpha}_\psi(y.z)\} and\ \vartheta_\psi(\rho.z) \leq max\ \{\vartheta_\psi(\rho.y)$

$for\ all\ \rho, y, z \in ℵ.$

$Example 4.2. Let\ X = \{0,1,2,3,4\} in\ which\ * is\ defined\ by\ Table$

.	0	1	2	3	4
0	0	1	2	3	4
1	0	0	0	0	1
2	0	3	0	3	4
3	0	1	2	0	1
4	0	1	0	0	0

Clearly (X,∗,0) is a KU-algebra. Define a cubic set ψ=($\tilde{\alpha}_\psi, \vartheta_\psi) in\ X\ as\ follows$

$\tilde{\alpha}_\psi(x) = \{[0.6,0.7], if\ x = 0\ [0.4,0.5], if\ x \in \{1,3\}, [0.1,0.3], if x \in \{2,4\},$

$\vartheta_\psi = \{0.1, if\ x = 0\ 0.3\ if\ x \in \{1,3\}, 0.6\ if\ x \in \{2,4\},$

By routine calculation it can be seen that the cubic set ψ=($\tilde{\alpha}_\psi, \vartheta_\psi) is\ a\ cubic\ GS - algebra\ of\ X.$

Lemma.4.3

Letψ=($\tilde{\alpha}_\psi, \vartheta_\psi) be\ a\ cubic\ GS\ algebra\ of\ X. if\ the\ inequality\ p * y \leq z\ holds\ in\ X, then\ \tilde{\alpha}_\psi(y) \leq rmin\{\tilde{\alpha}_\psi(p), \tilde{\alpha}_\psi(z)\} and\ \vartheta_\psi(y) \geq max\{\vartheta_\psi(p), \vartheta_\psi(z)\}.$

Proof: Assume that the inequality $p * y \leq z$ holds in X,then $p * y = 0\ and\ by\ (s_2)\ \tilde{\alpha}_\psi(p * z) \geq rmin\{\tilde{\alpha}_\psi(p * y), \tilde{\alpha}_\psi(y * z)\}, if\ we\ but\ z = 0$

Then $\tilde{\alpha}_\psi(p * 0) = \tilde{\alpha}_\psi(p) \geq rmin\{\tilde{\alpha}_\psi(p * y), \tilde{\alpha}_\psi(y * 0)\}$

$= rmin\{\tilde{\alpha}_\psi(p * y), \tilde{\alpha}_\psi(y)\} \dots \dots \dots \dots (1)$

But

$\tilde{\alpha}_\psi(p * z) \geq rmin\{\tilde{\alpha}_\psi(p * y)), \tilde{\alpha}_\psi(y * z)\}$

$= rmin\{\tilde{\alpha}_\psi(p * y), \tilde{\alpha}_\psi(y * z)\}$

$= rmin\{\tilde{\alpha}_\psi(0), \tilde{\alpha}_\psi(y * 0)\} = \tilde{\alpha}_\psi(y) \dots \dots \dots \dots (2)$

From (1) and (2), we get $\tilde{\alpha}_\psi(y) \leq rmin\{\tilde{\alpha}_\psi(p), \tilde{\alpha}_\psi(z)\}$.Similarly we can show that $\vartheta_\psi(y) \geq \{\vartheta_\psi(p), \vartheta_\psi(z)\}$.

this complet the proof.

Lemma4.4. IF ψ=($\tilde{\alpha}_\psi, \vartheta_\psi) be\ a\ cubic\ GS\ algebra\ of\ X$and if x≤ $y\ then\ \tilde{\alpha}_\psi(x) \geq \tilde{\alpha}_\psi(y) and\ \vartheta_\psi(x) \leq \vartheta_\psi(y)$

Proof: If x≤ y then $y * x = 0$.This together with $x * 0 = x$ and$\tilde{\alpha}_\psi(0) \geq \vartheta_\psi(y)$ also $\vartheta_\psi(0) \leq \vartheta_\psi(y)$,we get

$$\tilde{\alpha}_{\psi}(0*x) = \tilde{\alpha}_{\psi}(x) \geq rmin\{\tilde{\alpha}_{\psi}(0*(y*x)), \tilde{\alpha}_{\psi}(y)\}$$

$$= rmin\{\tilde{\alpha}_{\psi}(0*0), \tilde{\alpha}_{\psi}(y)\}$$

$$= rmin\{\tilde{\alpha}_{\psi}(0), \tilde{\alpha}_{\psi}(y)\} = \tilde{\alpha}_{\psi}(y).$$

Also, $\vartheta_{\psi}(0*x) = \vartheta_{\psi}(x) \geq rmin\{\vartheta_{\psi}(0*(y*x)), \vartheta_{\psi}(y)\} = rmin\{\vartheta_{\psi}(0*0), \vartheta_{\psi}(y)\}$

$$= rmin\{\vartheta_{\psi}(0), \vartheta_{\psi}(y)\} = \vartheta_{\psi}(y).$$

This completes the proof. ■

Let ψ=$(\tilde{\alpha}_{\psi}, \vartheta_{\psi}) and \vartheta = (\tilde{\alpha}_{\psi}, \vartheta_{\psi}) be\ two\ cubic\ sets\ in\ a\ KU-algebra\ X, then$

$$\psi \cap \vartheta = \{(x, rmin\{\tilde{\alpha}_{\psi}(x), \tilde{\alpha}_{\psi}(x)\}, \{\vartheta_{\psi}(x), \vartheta_{\psi}(x)\} : x \in X\}$$

$$=\{(x, \tilde{\alpha}_{\psi}(x) \cap \tilde{\alpha}_{\psi}(x), \vartheta_{\psi}(x) \cup \vartheta_{\psi}(x) : x \in X\}.$$

5.Image and Pre-image of cubic GS-algbra

In this section we will present some results on images and preimages of cubic GS-ideals in GS-algebras.

Definition5.1 Let $f\ be\ amapping\ from\ aset\ k\ to\ a\ set$ ḱ.If ψ=$(\tilde{\alpha}_{\psi}, \vartheta_{\psi}) is\ acubic\ set\ of\ \aleph, then\ the\ cubic\ subset\ \vartheta = (\tilde{\alpha}_{\vartheta}, \vartheta_{\vartheta}) of$ ḱ $is\ define\ by$

$$f(\tilde{\alpha}_{\psi})(y) = (\tilde{\alpha}_{\psi})(y) = \{rsup\ (\tilde{\alpha}_{\psi})(x)_{x \in f^{-1}(y)}\ if\ f^{-1}(y) = \{x \in X, f(x) = y\} \neq \emptyset\ 0,\ \ otherwise$$

$$f(\vartheta_{\psi})(y) = (\vartheta_{\psi})(y) = \{inf\ (\vartheta_{\psi})(x)_{x \in f^{-1}(y)}\ if\ f^{-1}(y) = \{x \in X, f(x) = y\} \neq \emptyset\ 1,\ \ otherwise$$

Is said to be the image of $\psi = (\tilde{\alpha}_{\psi}, \vartheta_{\psi}) under\ f$.

Similarly if $\mho = (\tilde{\alpha}_{\vartheta}, \vartheta_{\vartheta}) is\ a\ cubic\ subset\ of$ ḱ, $then\ the\ cubic\ subset\ \psi$=$\vartheta^{\circ} f\ in\ X(i.e., the\ cubic\ subset\ defined\ by\ \tilde{\alpha}_{\psi}(x) = \tilde{\alpha}_{\vartheta}(f(x)) for\ all\ x \in X) is\ called\ the\ preimage\ of$ **ϑ** under f.

Theorem.5.2. An epimorphism pre-image of a cubic GS-ieal is also cubic GS ideal.
Proof :Let $f: k \to$ ḱ $be\ an\ epimorphism\ mapping\ of\ GS-algebra. \vartheta = (\tilde{\alpha}_{\vartheta}, \vartheta_{\vartheta})$be a cubic T-ideal of ḱ and ψ=$(\tilde{\alpha}_{\psi}, \vartheta_{\psi}) be\ the\ pre-image\ of\ \vartheta\ under\ f, then \tilde{\alpha}_{\vartheta}(\rho) = \tilde{\alpha}_{\psi}(f(\rho)) and\ \vartheta_{\psi}(\rho) = \vartheta_{\vartheta}(f(\rho)) forany\ \rho \in k. then$

$$\tilde{\alpha}_{\vartheta}(0) = \tilde{\alpha}_{\vartheta}(f(0)) \geq \tilde{\alpha}_{\vartheta}(f(x)) = \tilde{\alpha}_{\vartheta}(x)$$

$$\vartheta_{\vartheta}(0) = \vartheta_{\vartheta}(f(0)) \geq \vartheta_{\vartheta}(f(x)) = \vartheta_{\vartheta}(x)$$

Now let p,y,z∈ $X, then$

$$\tilde{\alpha}_{\psi}(p*z) = \tilde{\alpha}_{\psi}(f(p*z)) = \tilde{\alpha}_{\psi}(f(p) *` f(z) \geq rmin\{\tilde{\alpha}_{\psi}\left(f(p) *` f(y)\right), \tilde{\alpha}_{\psi}(f(y) *` f(z))$$

$$=rmin\{\tilde{\alpha}_{\psi}(f(p*y)), \tilde{\alpha}_{\psi}(f(y*z))\}$$

$$.= rmin\{\tilde{\alpha}_{\psi}(p*y), \tilde{\alpha}_{\psi}(y*z)\}$$

$$\vartheta_{\psi}(p*z) = \vartheta_{\psi}(f(p*z)) = \vartheta_{\psi}(f(p) *` f(z)) \leq max\{\vartheta_{\psi}\left(f(p) *` f(y)\right), \vartheta_{\psi}(f(y) *` f(z))$$

$=max\{\vartheta_\psi(f(p*y)),\vartheta_\psi(f(y*z))\}$
$= max\{\vartheta_\psi(p*y),\vartheta_\psi(y*z)\}$.

This completes the proof.

Definition5.3. Acubic subset ψ =($\tilde{\alpha}_\psi, \vartheta_\psi$)of ℵ has sup and inf properties if for any subset T of ℵ ,there exist t,s∈ T such that $\tilde{\alpha}_\psi(t) = rsup_{t\in T}\tilde{\alpha}_\psi(t) and\ \vartheta_\psi(s) = inf_{t\in T}\vartheta_\psi(s)$.

Theorem5.4. Let $f{:}\,\mu \to Y$ *be anepimorphism between* $TM - algebra\ \mu\ and\ Y$.For every cubic T-ideal ψ=($\tilde{\alpha}_\psi, \vartheta_\psi$) in μ,then $f(\psi)$is cubic T-idealof Y.

Proof: Clear

6. Cartesian Product of Cubic GS-ideals

In this section we will provide some new definitions on Cartesian product of cubic GS-ideal in GS-algebras.

Definition 6.1 Let ψ_1=($\tilde{\alpha}_{\psi_1}, \vartheta_{\psi_1}$) and ψ_2=($\tilde{\alpha}_{\psi_2}, \vartheta_{\psi_2}$) be twocubic subsets of GS-algebras $X_1\ and\ X_2$,respectively. Then Cartesian product of cubic subsets $\psi_1\ and\ \psi_2$ is denoted by $\psi_1 \times \psi_2 = (\ \tilde{\alpha}_{\psi_1\times\psi_2}, \vartheta_{\psi_1\times\psi_2})$and is defined as

$$\tilde{\alpha}_{\psi_1\times\psi_2}(x,y) = rmin\{\tilde{\alpha}_{\psi_1}(x), \tilde{\alpha}_{\psi_2}(y)\},$$

$$\vartheta_{\psi_1\times\psi_2}(x,y) = \{\vartheta_{\psi_1}(x), \vartheta_{\psi_2}(y)\}\ ,$$

For all(x,y)∈ $X_1 \times X_2$.

Remark6.2 Let X and Y be GS-algebras. We define $*\,in\,X\times Y$ by (x,y) $*(u,v) = (x\ *u, y\ *v) for\ every(x,y),(u,v)$beliong to $X\times Y$,then clearly$(X\times Y,*,(0,0)) is\ a\ GS - algebra$.

Definition6.3. A cubic subset $\psi_1 \times \psi_2 == (\ \tilde{\alpha}_{\psi_1\times\psi_2}, \vartheta_{\psi_1\times\psi_2}) of X_1 \times X_2$ is called a cubic GS-subalgebra of $X_1 \times X_2$ if

$$(SP_1)\ \tilde{\alpha}_{\psi_1\times\psi_2}(0,0) \geq \tilde{\alpha}_{\psi_1\times\psi_2}(x,y) and\ \vartheta_{\psi_1\times\psi_2}(0,0) \leq \vartheta_{\psi_1\times\psi_2}(x,y),$$

$$(SP_2)\ \tilde{\alpha}_{\psi_1\times\psi_2}(x_1,y_1) \geq rmin\{\tilde{\alpha}_{\psi_1\times\psi_2}(x_1,y_1)*(x_2,y_2)),\tilde{\alpha}_{\psi_1\times\psi_2}(x_2,y_2)\},$$
$$\vartheta_{\psi_1\times\psi_2}(x_1,y_1) \leq\ max\ \{\vartheta_{\psi_1\times\psi_2}(x_1,y_1)*(x_2,y_2))),\vartheta_{\psi_1\times\psi_2}(x_2,y_2)\}.$$

$$(SP_3)\tilde{\alpha}_{\psi_1\times\psi_2}(x_1,y_1) \geq rmin\{\tilde{\alpha}_{\psi_1\times\psi_2}((x_1,y_1)*(x_2,y_2)),\tilde{\alpha}_{\psi_1\times\psi_2}(x_2,y_2)\},$$
$$\vartheta_{\psi_1\times\psi_2}(x_1,y_1) \leq\ max\ \{\vartheta_{\psi_1\times\psi_2}((x_1,y_1)*(x_2,y_2)),\vartheta_{\psi_1\times\psi_2}(x_2,y_2)\}$$

For all $(x_1,y_1),(x_2,y_2) \in X_1 \times X_2$.

Definition6.4A cubic subset $\psi_1 \times \psi_2 == (\ \tilde{\alpha}_{\psi_1\times\psi_2}, \vartheta_{\psi_1\times\psi_2}) of X_1 \times X_2$ is called a cubic GS-ideal of $X_1 \times X_2$ if

$$(SP_1)\ \tilde{\alpha}_{\psi_1\times\psi_2}(0,0) \geq \tilde{\alpha}_{\psi_1\times\psi_2}(x,y) and\ \vartheta_{\psi_1\times\psi_2}(0,0) \leq \vartheta_{\psi_1\times\psi_2}(x,y),$$

$$(SP_2)\ \tilde{\alpha}_{\psi_1\times\psi_2}\big((x_1,y_1)*(x_3,y_3)\big) \geq rmin\{\tilde{\alpha}_{\psi_1\times\psi_2}((x_1,y_1)*\big((x_2,y_2)\big),\tilde{\alpha}_{\psi_1\times\psi_2}((x_2,y_2)*(x_3,y_3)\} and$$

$$(\vartheta_{\psi_1\times\psi_2}\big((x_1,y_1)*(x_3,y_3)\big) \leq max\{\vartheta_{\psi_1\times\psi_2}(x_1,y_1))*\big((x_2,y_2)\big),\vartheta_{\psi_1\times\psi_2}((x_2,y_2)*(x_3,y_3))\},$$

For all $(x_1,y_1),(x_2,y_2)\ (x_3,y_3) \in X_1 \times X_2$.

(SP_3) $\tilde{\alpha}_{\psi_1\times\psi_2}\big((x_1,y_1).(x_3,y_3)\big)$
$\geq rmin\{\tilde{\alpha}_{\psi_1\times\psi_2}((x_1,y_1).\big((x_2,y_2)\big),\tilde{\alpha}_{\psi_1\times\psi_2}((x_2,y_2).(x_3,y_3)\}$

and$(\vartheta_{\psi_1\times\psi_2}\big((x_1,y_1).(x_3,y_3)\big)\leq max\{\vartheta_{\psi_1\times\psi_2}(x_1,y_1)).\big((x_2,y_2)\big),\vartheta_{\psi_1\times\psi_2}((x_2,y_2).(x_3,y_3))\},$

Theorem.6.5. Let ψ_1=($\tilde{\alpha}_{\psi_1},\vartheta_{\psi_1}$) and ψ_2=($\tilde{\alpha}_{\psi_2},\vartheta_{\psi_2}$) be twocubic subsets of GS-algebras X_1 *and* X_2,respectively.then $\psi_1\times\psi_2=\big(\tilde{\alpha}_{\psi_1\times\psi_2},\vartheta_{\psi_1\times\psi_2}\big)$*is acubic GS − subalgebra ofGS − algebra* $X_1\times X_2$.

Proof: For any $\in X_1\times X_2$,

$$\tilde{\alpha}_{\psi_1\times\psi_2}(0,0)=rmin\{\tilde{\alpha}_{\psi_1}(0),\tilde{\alpha}_{\psi_2}(0)\}\geq rmin\{\tilde{\alpha}_{\psi_1}(x),\tilde{\alpha}_{\psi_2}(y)\}=\tilde{\alpha}_{\psi_1\times\psi_2}(x,y),$$

$$\vartheta_{\psi_2}(0)\}\leq max\ \{\vartheta_{\psi_1}(x),\vartheta_{\psi_2}(y)\}=\vartheta_{\psi_1\times\psi_2}(x,y).\vartheta_{\psi_1\times\psi_2}(0,0)=max\ \{\vartheta_{\psi_1}(0),$$

For any $(x_1,y_1),(x_2,y_2)\in X_1\times X_2.Then$

$$\tilde{\alpha}_{\psi_1\times\psi_2}=rmin\{\tilde{\alpha}_{\psi_1}(x_1),\tilde{\alpha}_{\psi_2}(z_1)\}\geq rmin\{\{\tilde{\alpha}_{\psi_1}(x_1*x_2),\tilde{\alpha}_{\psi_2}(x_2)\},$$

$$rmin\{\tilde{\alpha}_{\psi_2}(y_1*y_2),\tilde{\alpha}_{\psi_1}(y_2)\}$$
$$=rmin\{rmin\{\tilde{\alpha}_{\psi_1}(x_1*x_2),\tilde{\alpha}_{\psi_2}(y_1*y_2)\},rmin\{\tilde{\alpha}_{\psi_1}(x_2),\tilde{\alpha}_{\psi_2}(y_2)\}$$

$$=rmin\{\tilde{\alpha}_{\psi_1\times\psi_2}(x_1*x_2,y_1*y_2),\tilde{\alpha}_{\psi_1\times\psi_2}(x_2,y_2)\},$$

$$\vartheta_{\psi_1\times\psi_2}(x_1,y_1)=\{\vartheta_{\psi_1}(x_1),\vartheta_{\psi_2}(y_1)\}$$
$$\leq max[\{\vartheta_{\psi_1}(x_1*x_2),\vartheta_{\psi_1}(x_2)\},\{\vartheta_{\psi_2}(y_1*y_2),\vartheta_{\psi_2}(y_2)\}$$
$$=max\ \{\{\vartheta_{\psi_1}(x_1*x_2),\vartheta_{\psi_2}(y_1*y_2)\},\{\vartheta_{\psi_1}(x_2),\vartheta_{\psi_2}(y_2)\}$$
$$=\{\vartheta_{\psi_1\times\psi_2}(x_1*x_2,y_1*y_2),\vartheta_{\psi_1\times\psi_2}(x_2,y_2)\}\}$$

$$\leq\{\vartheta_{\psi_1\times\psi_2}(x_1*y_1)(x_2*y_2)),\vartheta_{\psi_1\times\psi_2}(x_2*y_2)$$

Theorem.6.6. Let ψ_1=($\tilde{\alpha}_{\psi_1},\vartheta_{\psi_1}$) and ψ_2=($\tilde{\alpha}_{\psi_2},\vartheta_{\psi_2}$) be two cubic ideal of GS-algebras X_1 *and* X_2,respectively.then$\psi_1\times\psi_2=\big(\tilde{\alpha}_{\psi_1\times\psi_2},\vartheta_{\psi_1\times\psi_2}\big)$*is a cubic GS − ideal ofGS − algebra* $X_1\times X_2$

Proof: For any (x,y)$\in X_1\times X_2$,

$$\tilde{\alpha}_{\psi_1\times\psi_2}(0,0)=rmin\{\tilde{\alpha}_{\psi_1}(0),\tilde{\alpha}_{\psi_2}(0)\}\geq rmin\{\tilde{\alpha}_{\psi_1}(x),\tilde{\alpha}_{\psi_2}(y)\}=\tilde{\alpha}_{\psi_1\times\psi_2}(x,y),$$

$$\vartheta_{\psi_1\times\psi_2}(0,0)=\{\vartheta_{\psi_1}(0),\vartheta_{\psi_2}(0)\}\leq\{\vartheta_{\psi_1}(x),\vartheta_{\psi_2}(y)\}=\vartheta_{\psi_1\times\psi_2}(x,y).$$

Now for any $(x_1,y_1),(x_2,y_2),(x_3,y_3)\in X_1\times X_2$,

$,y_3))\tilde{\alpha}_{\psi_1\times\psi_2}((x_1*y_1)*(x_3$

$=\tilde{\alpha}_{\psi_1\times\psi_2}\ (x_1*x_3,y_1*y_3)$

$$=rmin\{\tilde{\alpha}_{\psi_1}(x_1*x_3),\tilde{\alpha}_{\psi_2}(y_1*y_3)\}\geq rmin\Big\{rmin\{\tilde{\alpha}_{\psi_1}\big(x_1*(x_2*x_3)\big),\tilde{\alpha}_{\psi_1}(x_2)\}\Big\}$$
$$=rmin\{\tilde{\alpha}_{\psi_2}\big(y_1*(y_2*y_3)\big),\tilde{\alpha}_{\psi_2}(y_2)\}$$

$$=rmin\{rmin\{\tilde{\alpha}_{\psi_1}\big(x_1*(x_2*x_3)\big),\tilde{\alpha}_{\psi_2}\big(y_1*(y_2*y_3)\big)\},rmin\{\tilde{\alpha}_{\psi_1}(x_2),\tilde{\alpha}_{\psi_2}(y_2)\}$$

$$,\tilde{\alpha}_{\psi_1\times\psi_2}(x_2,y_2)\}=rmin\{\tilde{\alpha}_{\psi_1\times\psi_2}\big(x_1*(x_2*x_3)\big),(y_1*(y_2*y_3)\}$$

$$\geq rmin\{\tilde{\alpha}_{\psi_1\times\psi_2}((x_1,y_1)*(x_2,y_2))*(x_3,y_3)),\tilde{\alpha}_{\psi_1\times\psi_2}(x_2,y_2)\},$$

$$(x_3,y_3))=\vartheta_{\psi_1\times\psi_2}(x_1*x_3,y_1*y_3)\vartheta_{\psi_1\times\psi_2}((x_1,y_1)*$$

$$= max\ \{\vartheta_{\psi_1}(x_1 * x_3), \vartheta_{\psi_2}(y_1 * y_3)\} \leq max\ \{max\{\vartheta_{\psi_1}(x_1 * (x_2 * x_3)), \vartheta_{\psi_1}(x_2)\}$$
$$= rmin\{\tilde{\alpha}_{\psi_2}(y_1 * (y_2 * y_3)), \tilde{\alpha}_{\psi_2}(y_2)\}$$

$$= rmin\{rmin\{\vartheta_{\psi_1}(x_1 * (x_2 * x_3)), \vartheta_{\psi_2}(y_1 * (y_2 * y_3))\}, rmin\{\vartheta_{\psi_1}(x_2), \vartheta_{\psi_2}(y_2)\}$$

$$,\vartheta_{\psi_1\times\psi_2}(x_2, y_2)\}= rmin\{\vartheta_{\psi_1\times\psi_2}(x_1 * (x_2 * x_3)), (y_1 * (y_2 * y_3)\}$$

$$\geq rmin\{\vartheta_{\psi_1\times\psi_2}((x_1, y_1) * (x_2, y_2)) * (x_3, y_3)), \vartheta_{\psi_1\times\psi_2}(x_2, y_2)\},$$

$$(x_3, y_3)) = \vartheta_{\psi_1\times\psi_2}(x_1.x_3, y_1.y_3)\vartheta_{\psi_1\times\psi_2}((x_1, y_1).$$

$$= max\ \{\vartheta_{\psi_1}(x_1.x_3), \vartheta_{\psi_2}(y_1.y_3)\} \leq max\ \{max\{\vartheta_{\psi_1}(x_1.(x_2.x_3)), \vartheta_{\psi_1}(x_2)\}$$
$$= rmin\{\tilde{\alpha}_{\psi_2}(y_1.(y_2.y_3)), \tilde{\alpha}_{\psi_2}(y_2)\}$$

$$= rmin\{rmin\{\vartheta_{\psi_1}(x_1.(x_2.x_3)), \vartheta_{\psi_2}(y_1.(y_2.y_3))\}, rmin\{\vartheta_{\psi_1}(x_2), \vartheta_{\psi_2}(y_2)\}$$

$$,\vartheta_{\psi_1\times\psi_2}(x_2, y_2)\}= rmin\{\vartheta_{\psi_1\times\psi_2}(x_1.(x_2.x_3)), (y_1.(y_2.y_3)\}$$

$$\geq rmin\{\vartheta_{\psi_1\times\psi_2}((x_1, y_1).(x_2, y_2)).(x_3, y_3)), \vartheta_{\psi_1\times\psi_2}(x_2, y_2)$$

Lemma.6.7.If $\psi_1 \times \psi_2 = (\tilde{\alpha}_{\psi_1\times\psi_2}, \vartheta_{\psi_1\times\psi_2})$ $is\ a\ cubic\ GS - ideal\ of GS - algebra\ X_1 \times X_2$ and if $(x_1, y_1) \leq (x_2, y_2)$,we have $\tilde{\alpha}_{\psi_1\times\psi_2}(x_2, y_2) \leq \tilde{\alpha}_{\psi_1\times\psi_2}(x_1, y_1) and\ \ \vartheta_{\psi_1\times\psi_2}(x_2, y_2) \geq \vartheta_{\psi_1\times\psi_2}(x_1, y_1), for\ all$

$\in X_1 \times X_2$.$(x_1, y_1), (x_2, y_2)$

Proof:Let $(x_1, y_1), (x_2, y_2) \in X_1 \times X_2$,such that $(x_1, y_1) \leq (x_2, y_2) \Rightarrow (x_2, y_2) * (x_1, y_1) = (0,0) * (x_1, y_1) = (x_1, y_1)$ and $\tilde{\alpha}_{\psi_1\times\psi_2}(x_2, y_2) \leq \tilde{\alpha}_{\psi_1\times\psi_2}(0,0)$

Also $\vartheta_{\psi_1\times\psi_2}(x_2, y_2) \geq \vartheta_{\psi_1\times\psi_2}(0,0)$ .Concsider

$(0,0)* (x_2, y_2)) = \tilde{\alpha}_{\psi_1\times\psi_2}(x_1, y_1)\vartheta_{\psi_1\times\psi_2}$

$$\geq rmin\{\tilde{\alpha}_{\psi_1\times\psi_2}((0,0) * (x_2, y_2) * (x_1, y_1)), \tilde{\alpha}_{\psi_1\times\psi_2}(x_2, y_2)\}$$

$$=rmin\{\tilde{\alpha}_{\psi_1\times\psi_2}((0,0) * (0,0)), \tilde{\alpha}_{\psi_1\times\psi_2}(x_2, y_2)\}$$

$$= rmin\{\tilde{\alpha}_{\psi_1\times\psi_2}(0,0), \tilde{\alpha}_{\psi_1\times\psi_2}(x_2, y_2)\} = \tilde{\alpha}_{\psi_1\times\psi_2}(x_2, y_2)$$

$(0,0)* (x_1, y_1)) = \vartheta_{\psi_1\times\psi_2}(x_1, y_1)\vartheta_{\psi_1\times\psi_2}$

$(0,0)* (x_2, y_2) * (x_1, y_1)), \vartheta_{\psi_1\times\psi_2}(x_2, y_2)\}\leq max\ \{\vartheta_{\psi_1\times\psi_2}$

$(0,0)* (0,0)), \vartheta_{\psi_1\times\psi_2}(x_2, y_2)\}= max\ \{\vartheta_{\psi_1\times\psi_2}$

$(0,0), \vartheta_{\psi_1\times\psi_2}(x_2, y_2)\}= max\ \{\vartheta_{\psi_1\times\psi_2}$

$$= \vartheta_{\psi_1\times\psi_2}(x_2, y_2)\}$$

This shows that $\tilde{\alpha}_{\psi_1\times\psi_2}(x_2, y_2) \leq \tilde{\alpha}_{\psi_1\times\psi_2}(x_1, y_1) and\ \ \vartheta_{\psi_1\times\psi_2}(x_2, y_2) \geq \vartheta_{\psi_1\times\psi_2}(x_1, y_1), for\ all (x_1, y_1), (x_2, y_2) \in X_1 \times X_2$.

Lemma 6.8. If $\psi_1 \times \psi_2 = (\tilde{\alpha}_{\psi_1\times\psi_2}, \vartheta_{\psi_1\times\psi_2})$ $is\ a\ cubic\ GS - ideal\ of GS - algebra\ X_1 \times X_2$and if $(x_1, y_1) * (x_2, y_2) \leq (x_3, y_3) hold\ in\ X_1 \times X_2, then\ we\ have\ \tilde{\alpha}_{\psi_1\times\psi_2}(x_2, y_2) \geq rmin\{\tilde{\alpha}_{\psi_1\times\psi_2}(x_1, y_1), \tilde{\alpha}_{\psi_1\times\psi_2}(x_3, y_3)\} and\ \ \vartheta_{\psi_1\times\psi_2}(x_2, y_2) \leq max\{\vartheta_{\psi_1\times\psi_2}(x_1, y_1), \vartheta_{\psi_1\times\psi_2}(x_3, y_3)\}$,For all $(x_1, y_1), (x_2, y_2)\ (x_3, y_3) \in X_1 \times X_2$.

Proof:Let $(x_1, y_1), (x_2, y_2)$ $(x_3, y_3) \in X_1 \times X_2$and let $(x_1, y_1) * (x_2, y_2) \leq (x_3, y_3) hold\ in\ X_1 \times X_2, then$

$^*(x_1, y_1) * (x_2, y_2)$=(0,0).$(x_3, y_3)$

Now for (x_3, y_3)=(0,0) and from (SP_2)

$$\tilde{\alpha}_{\psi_1\times\psi_2}((x_3, y_3) * (x_2, y_2))$$

$$\geq rmin\{\tilde{\alpha}_{\psi_1\times\psi_2}\big((x_3, y_3)\big) * (x_1, y_1) * (x_2, y_2)), \tilde{\alpha}_{\psi_1\times\psi_2}(x_1, y_1)\},$$

$$\tilde{\alpha}_{\psi_1\times\psi_2}\big((0,0) * (x_2, y_2)\big) = \tilde{\alpha}_{\psi_1\times\psi_2}(x_2, y_2)$$

$$\geq rmin\{\tilde{\alpha}_{\psi_1\times\psi_2}(x_1, y_1) * (x_2, y_2)), \tilde{\alpha}_{\psi_1\times\psi_2}(x_1, y_1)\}$$

$$\geq rmin\{rmin\{\tilde{\alpha}_{\psi_1\times\psi_2}(x_1, y_1) * (x_3, y_3)(x_2, y_2))), \tilde{\alpha}_{\psi_1\times\psi_2}(x_3, y_3)\}, \tilde{\alpha}_{\psi_1\times\psi_2}(x_1, y_1)\}$$

$$= rmin\{rmin\{\tilde{\alpha}_{\psi_1\times\psi_2}(x_3, y_3) * \big((x_1, y_1) * (x_2, y_2)\big)), \tilde{\alpha}_{\psi_1\times\psi_2}(x_3, y_3)\}, \tilde{\alpha}_{\psi_1\times\psi_2}(x_1, y_1)\},$$

$$= rmin\big\{rmin\{\tilde{\alpha}_{\psi_1\times\psi_2}(0,0), \tilde{\alpha}_{\psi_1\times\psi_2}(x_3, y_3)\big\}, \tilde{\alpha}_{\psi_1\times\psi_2}(x_1, y_1)$$

$$rmin\big\{\tilde{\alpha}_{\psi_1\times\psi_2}(x_3, y_3), \tilde{\alpha}_{\psi_1\times\psi_2}(x_1, y_1)\big\}$$

$$=rmin\{\tilde{\alpha}_{\psi_1\times\psi_2}(x_1, y_1), \tilde{\alpha}_{\psi_1\times\psi_2}(x_3, y_3)\},$$

And from (SP_3)

$$\vartheta_{\psi_1\times\psi_2}\big((0,0) * (x_2, y_2)\big) = \vartheta_{\psi_1\times\psi_2}(x_2, y_2)$$

$$\leq max\ \{\{\vartheta_{\psi_1\times\psi_2}(x_1, y_1) * (x_2, y_2)), \vartheta_{\psi_1\times\psi_2}(x_1, y_1)\}$$

$$\leq max\{max\{\vartheta_{\psi_1\times\psi_2}(x_1, y_1) * (x_3, y_3)(x_2, y_2))), \vartheta_{\psi_1\times\psi_2}(x_3, y_3)\}, \vartheta_{\psi_1\times\psi_2}(x_1, y_1)\}$$

$$= max\{max\{\vartheta_{\psi_1\times\psi_2}(x_3, y_3) * \big((x_1, y_1) * (x_2, y_2)\big)), \vartheta_{\psi_1\times\psi_2}(x_3, y_3)\}, \vartheta_{\psi_1\times\psi_2}(x_1, y_1)\},$$

$$= max\big\{max\{\vartheta_{\psi_1\times\psi_2}(0,0), \vartheta_{\psi_1\times\psi_2}(x_3, y_3)\big\}, \vartheta_{\psi_1\times\psi_2}(x_1, y_1)$$

$$= max\big\{\vartheta_{\psi_1\times\psi_2}(x_3, y_3), \vartheta_{\psi_1\times\psi_2}(x_1, y_1)\big\}$$

$$=max\{\vartheta_{\psi_1\times\psi_2}(x_1, y_1), \vartheta_{\psi_1\times\psi_2}(x_3, y_3)\},$$

This completes the proof

4. Conclusion

This study introduces the concept of cubic ideals in GS-algebras and explores their properties, including their behavior under homomorphisms and Cartesian products. Key findings include the definition and characterization of cubic sub-algebras, relationships between level subsets and cubic sub-algebras, and the behavior of images and pre-images of cubic GS-ideals. It was demonstrated that the Cartesian product of two cubic ideals in GS-algebras also forms a cubic ideal. These results extend the theoretical framework of GS-algebras and provide a foundation for further research into their algebraic properties and potential applications.

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